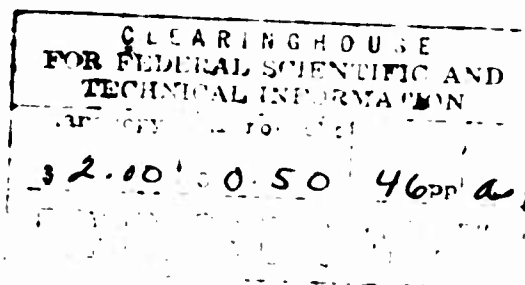


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ENGINEERING GRAPHICS SEMINAR
FOUR-DIMENSIONAL DESCRIPTIVE GEOMETRY
METRIC PROBLEMS: ANGLES - DESCRIPTIVE SOLUTION

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ABSTRACT

The paper presents the descriptive geometry solution of the determination of the angles between geometric elements in a four-dimensional euclidean space.

Incorporated in these solutions are the discussions of some problems of intersections between two geometric elements. These are: intersection of a line and a plane, belonging to the same 3-D space; intersection of two planes belonging to distinct 3-D spaces; intersection of a line and a 3-D space; intersection of a plane and a 3-D space.

Problem No. 1

"To determine the true value of the angle of two lines".

Preliminary discussion: if the two lines are non-concurrent, through a point of one draw a line parallel to the other and determine the angle of the two concurrent lines.

Solution

Consider a 3-D space that belongs to the plane determined by the two concurrent lines, apply a rotation of the 3-D space¹⁾ until it is superimposed on one of the 3-D spaces of the 4-D system of reference, and use a three-dimensional method of descriptive geometry to determine the true value of the angle.

To determine a 3-D space that belongs to the plane (abc) find the points of intersection of plane (abc) with the planes π_1 , π_2 , π_3 of the 4-D system of reference. (Figure 2)

¹⁾ See the author's Special Report No. 4, Engineering Graphics Seminar, Department of Graphics and Engineering Drawing, Princeton University, March, 1965.

2.

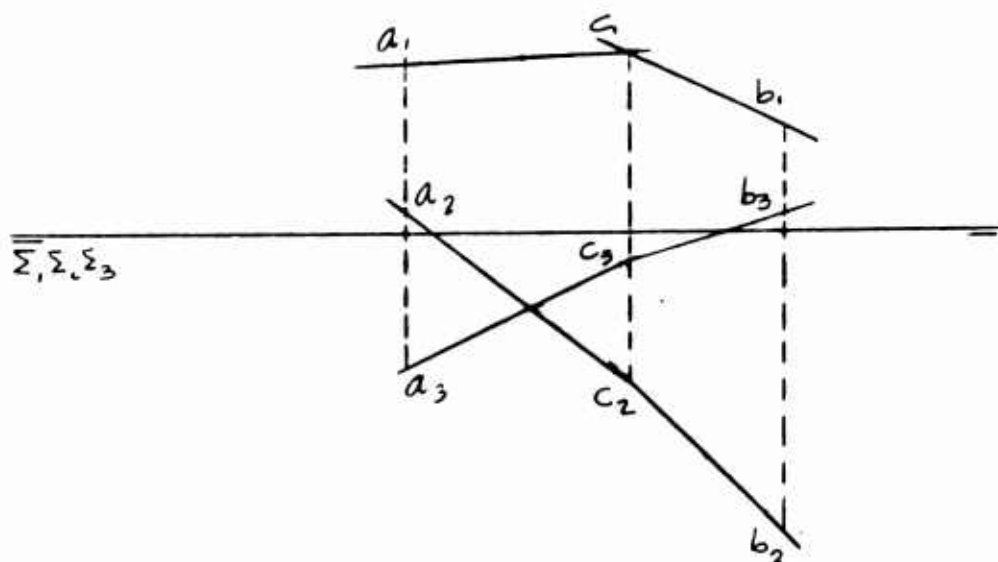


Figure 1

Next, using τ_1, τ_2, τ_3 as axes of reference, determine the position of (a), (b), and (c). See pictorial view in Figure 3 and constructions in the orthographic projection in Figure 4.

To determine the superimposition of 3-D space T on the 3-D space Σ_3 of the 4-D system of reference, consult our Special Report No. 4²⁾. The graphical constructions are shown in Figures 5, 6, and 7.

2) Engineering Graphics Seminar, Technical Seminar Series, Department of Graphics and Engineering Drawing, Princeton University.

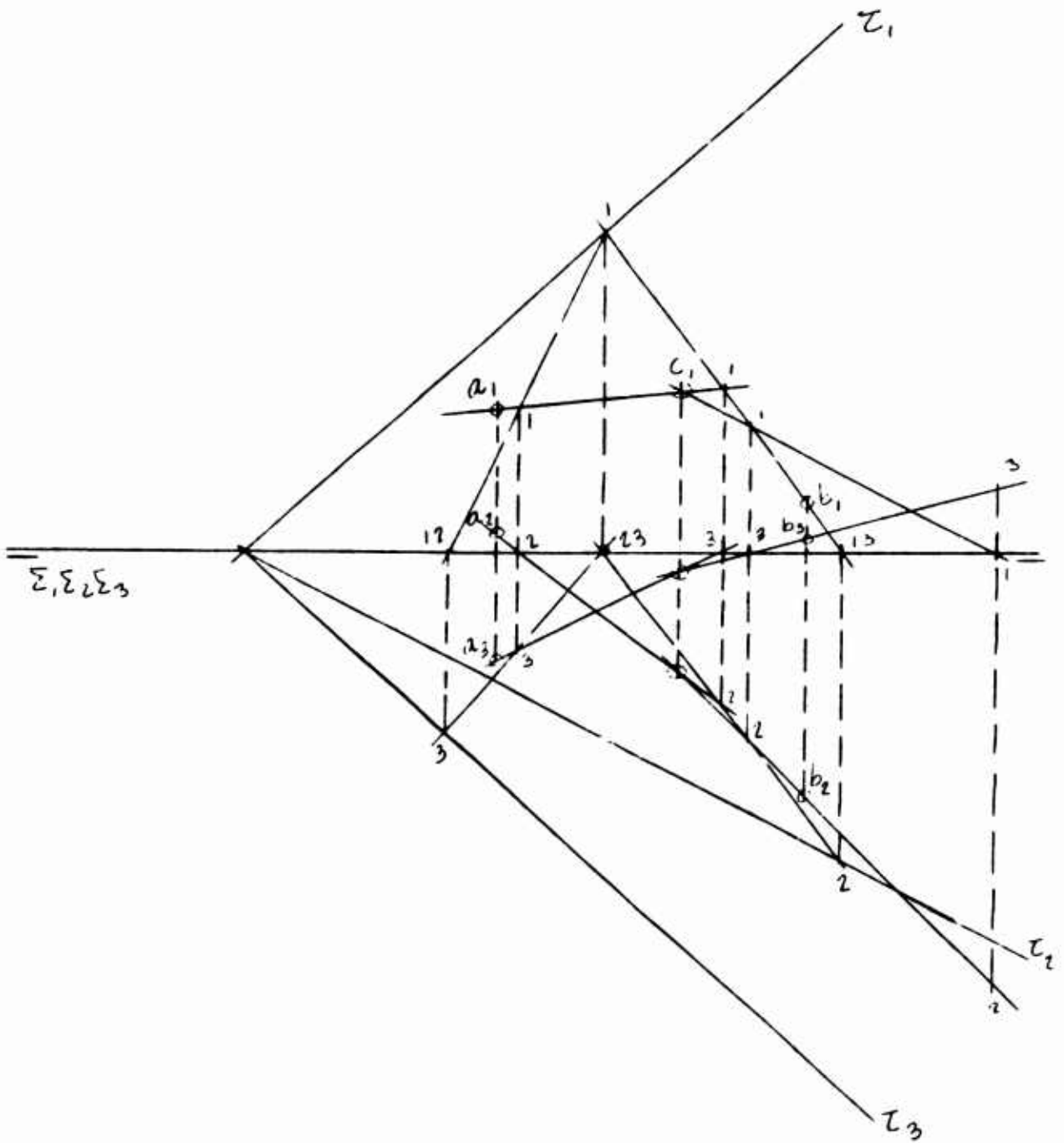


Figure 2

4.

As seen in Figure 7, the line (ob'') represents the trace \mathcal{T}_3 of the 3-D space \mathcal{T} . We shall proceed by determining the positions of (a') and (c') on that line (Figure 8). The true lengths of (oa') and (oc') are obtained from Figure 5.

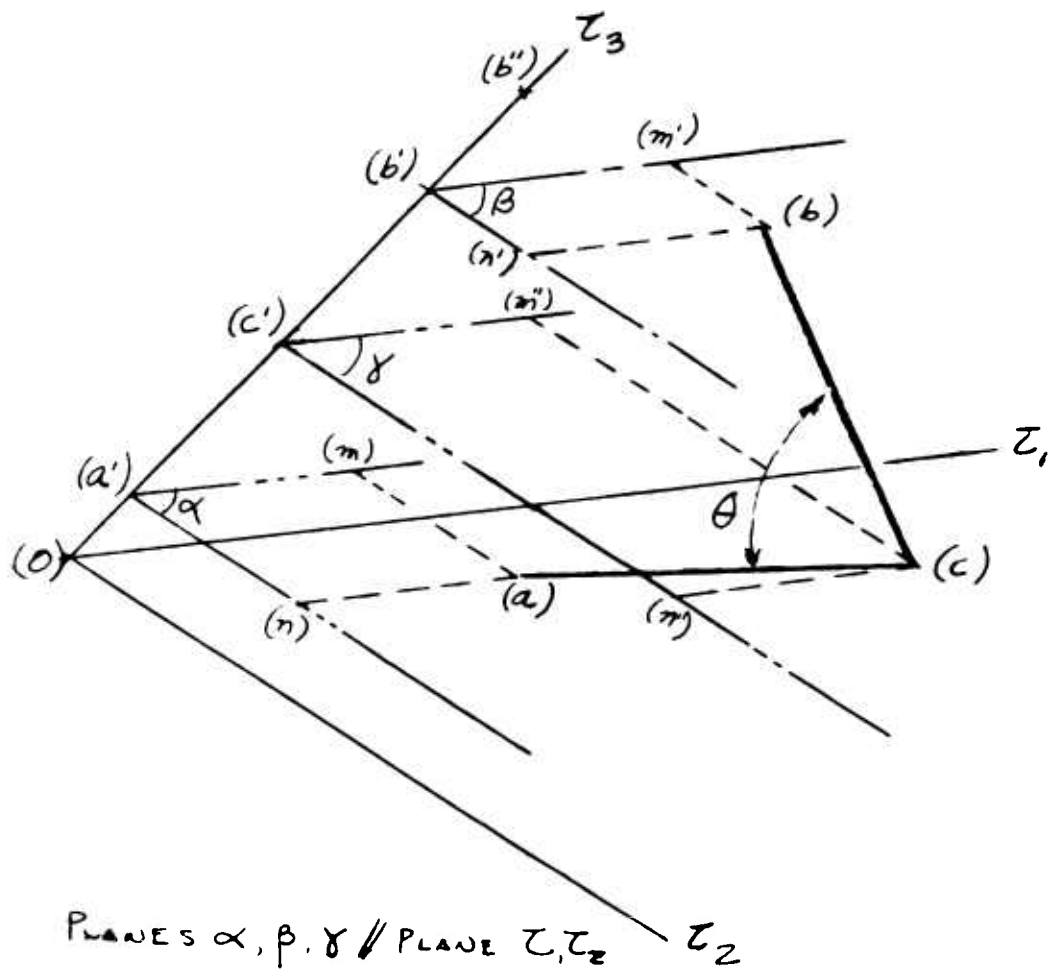


Figure 3

5.

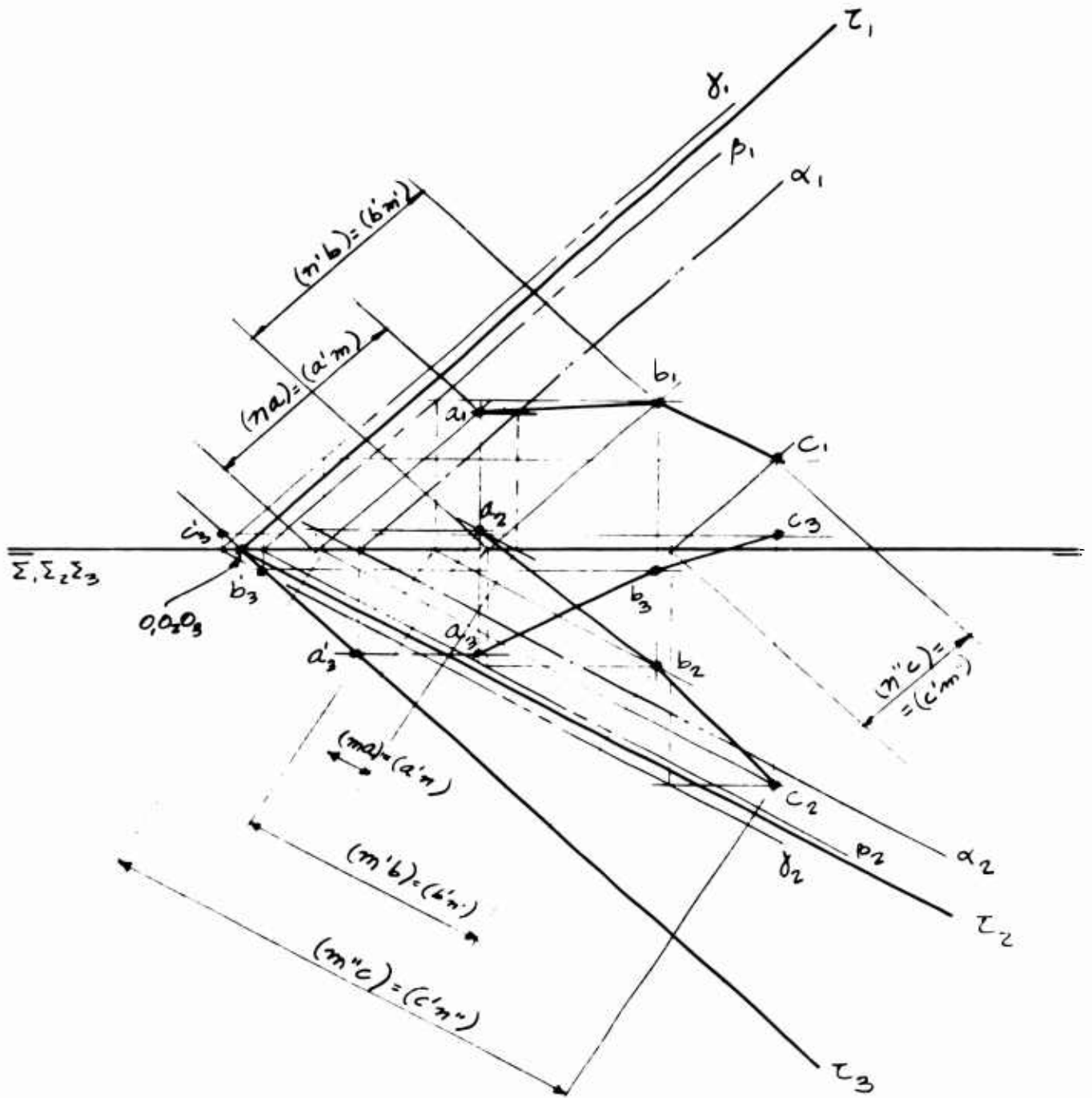
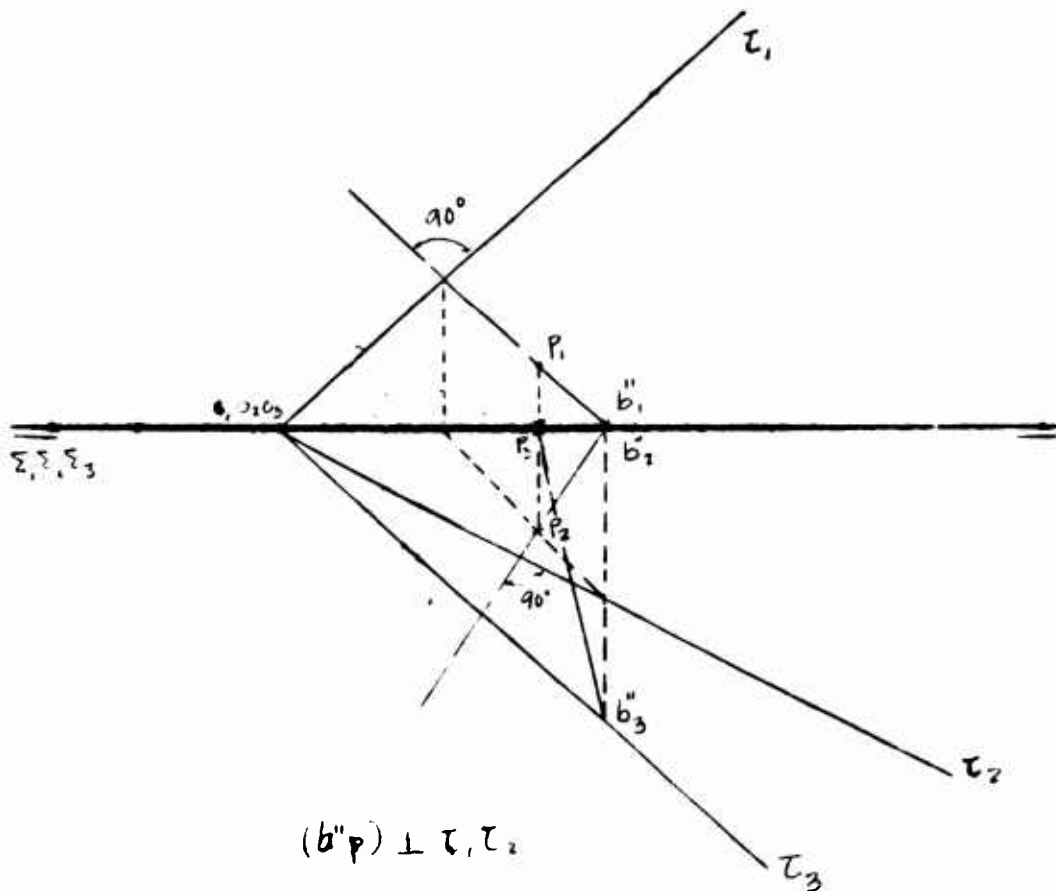


Figure 4

6.



$(b''p) \perp \tau, \tau_2$
 (p) IS THE FOOT OF THE PERPENDICULAR

Figure 5

Next, draw $(a'm)$, $(b'm')$, $(c'm'')$ parallel to τ_1 , $(a'n)$, $(b'n')$, $(c'n'')$ parallel to τ_2 , marking its respective lengths obtained from Figure 5. Then, through (m) and (n) draw parallels to τ_2 and τ_1 , respectively, to determine (a) . Use similar procedures to obtain (b) and (c) , working with points (m') , (n') , (m'') , and (n'') .
 Figure 9.

7.

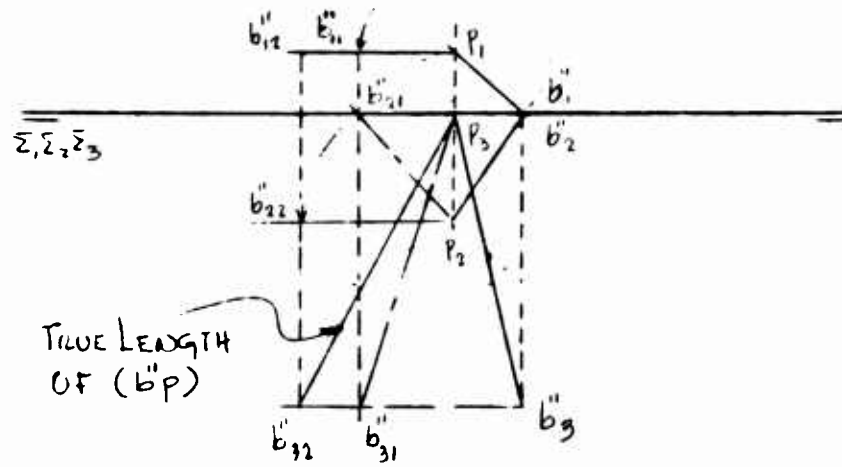


Figure 6

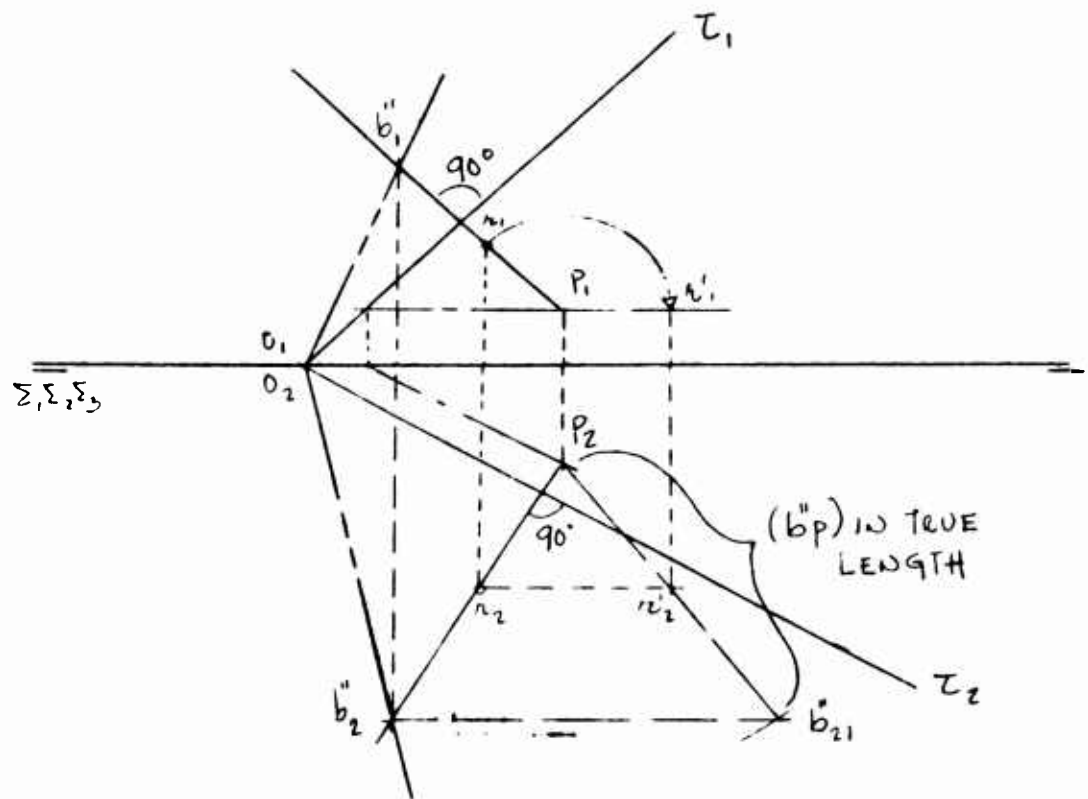


FIGURE 7

8.

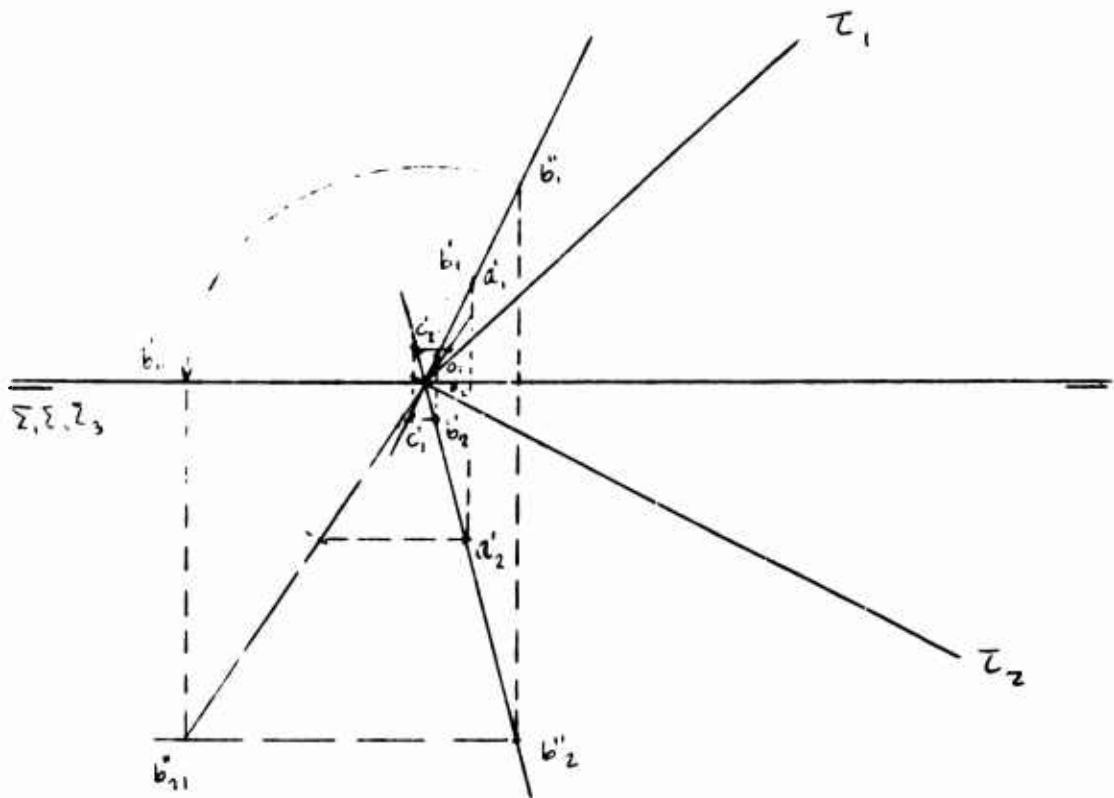


Figure 8

Thus, we obtained the projections of the three points (a), (b), (c), of the 3-D space \mathcal{T} , when superimposed on the 3-D space Σ_3 . To determine the true value of the angle $(a\hat{c}b)$, use a method of the three-dimensional descriptive geometry. (Figure 10).

9.

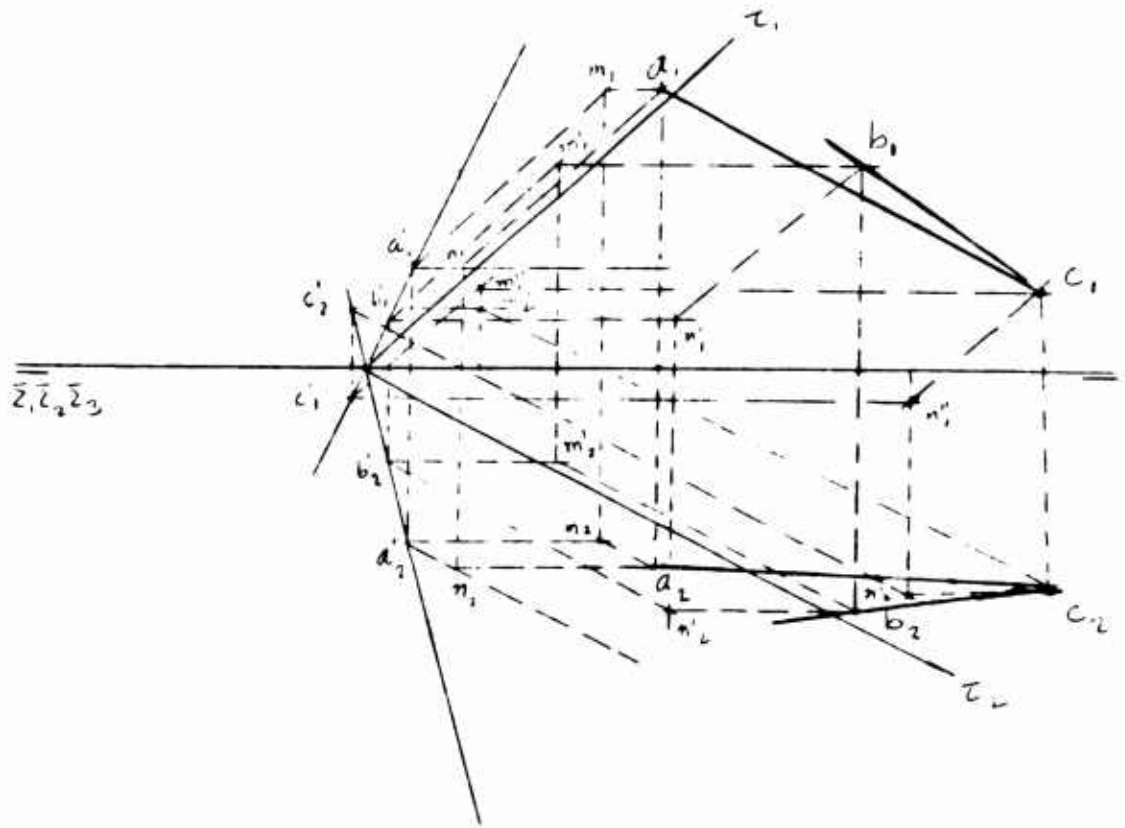


Figure 9

10.

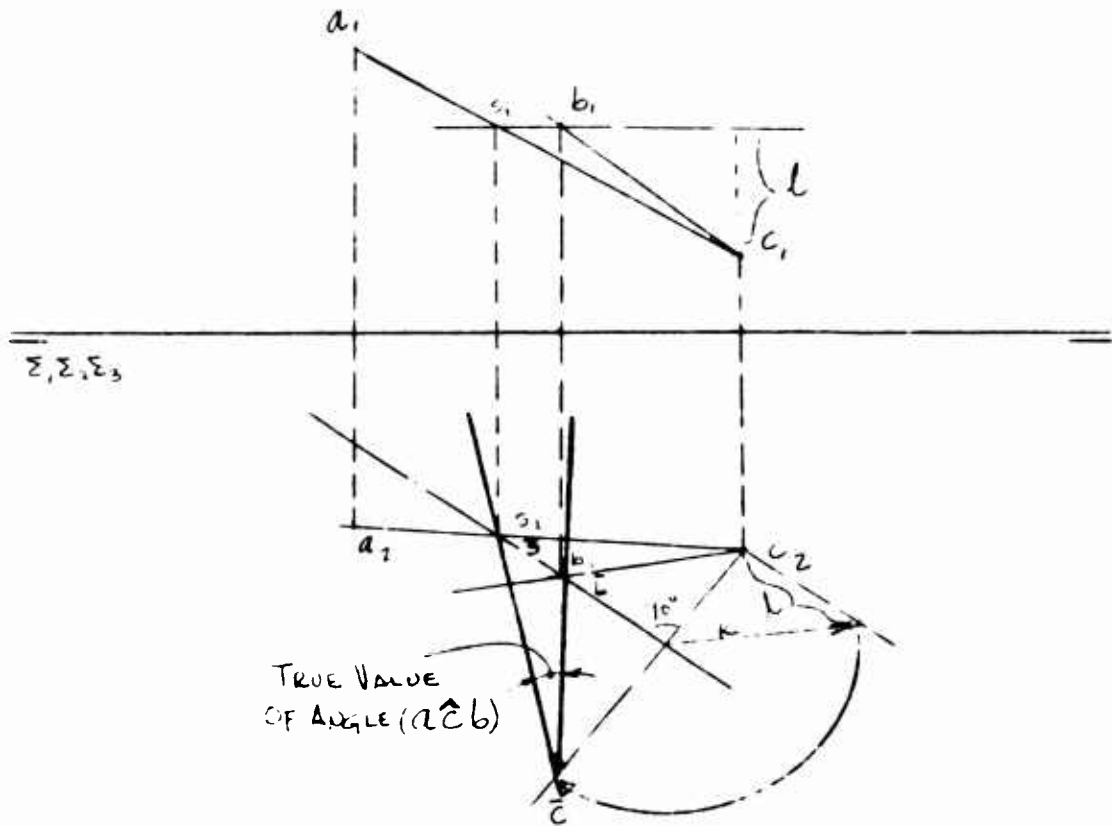


Figure 10

Problem No. 2

"To determine the true value of the angle of a line and a plane".

Preliminary discussion: If the line and the plane are non-concurrent, (belonging to two distinct 3-D spaces),

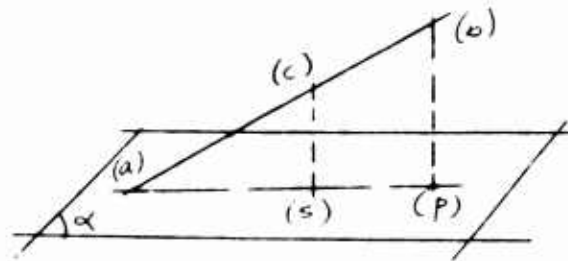
11.

through a point of the plane draw a line parallel to the given one and determine its angle with the plane.

Solution

The angle of a line with a plane is equal to the angle of the line with its orthogonal projection on the plane.

Thus we should find the point (a), intersection of the line and the plane and through a point (b), of the line draw a perpendicular to the plane and determine its foot, point (p). The angle $(b\hat{a}p)$ is the angle of (ab) with the plane. (Figure 11).



$$\left. \begin{array}{l} (ab) \times \alpha \Rightarrow (a) \\ (bp) \perp \alpha \end{array} \right\} (b\hat{a}p) = \widehat{(ab)}, \alpha$$

$$\begin{array}{l} \text{IF } (a) \text{ IS INACCESSIBLE} \\ \widehat{(cb)}, (sp) = \widehat{(ab)}, \alpha \\ \text{WHERE } (cs) \perp \alpha \end{array}$$

Figure 11

12.

The general proposition of the problem gives a line $(b'c')$ and a plane α . In Figure 12, the plane α is given by two concurrent lines (R) and (T).

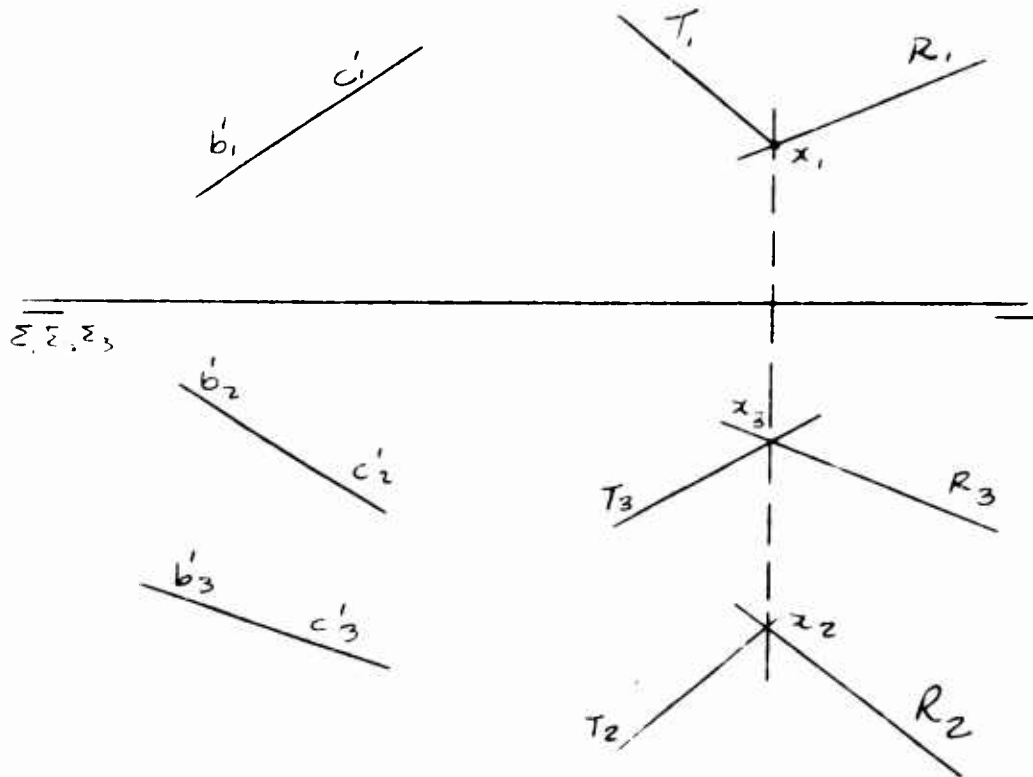


Figure 12

Without investigating if they belong to the same 3-D space, take a point (a) in (R) and draw (ab) parallel to $(b'c')$. (Figure 13).

13.

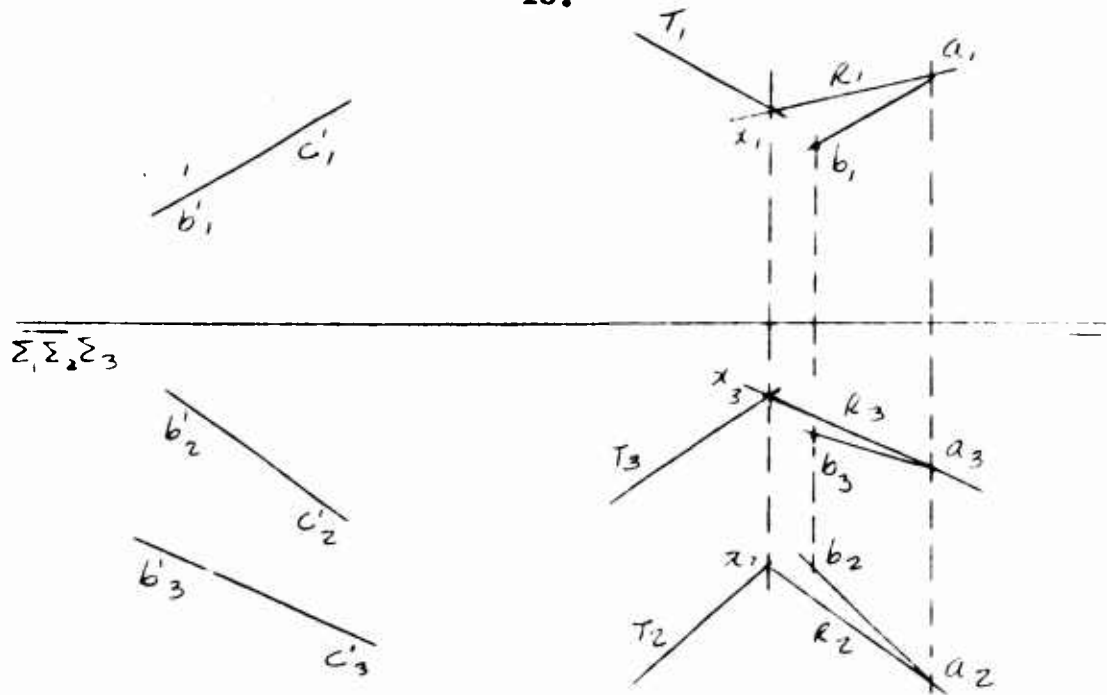


Figure 13

Consider the 3-D space $(R, T - b)$ to proceed with the solution.

Since through (b) we have to draw perpendicular to plane $(R, T) \equiv \alpha$, it will be necessary to apply a descriptive method so that we may properly operate within the 3-D space $(R, T - b)$. The objective should be to place the plane α parallel to one of the 3-D spaces of the 4-D system of reference, as indicated in Figure 14. Here, the plane α is parallel to Σ_3 and belongs to the 3-D space T .

14.

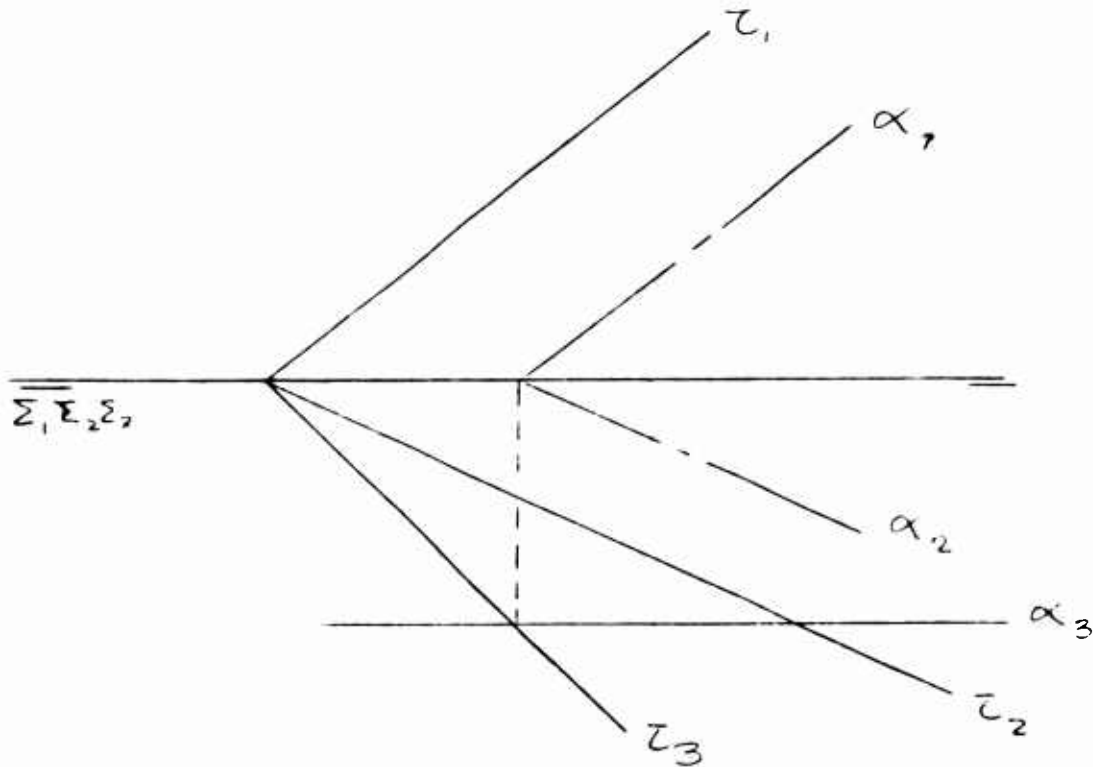


Figure 14

The graphical constructions pertinent to the application of the descriptive method are outlined in other papers³⁾ and we shall not repeat them here.

Therefore, let us admit the 3-D space $T \in (R, T - b)$ of Figure 15, given by its traces, and the plane α and point (b) belonging to it. In the plane α we consider

3) Descriptive Geometry of Four Dimensions, Engineering Graphics Seminar, Technical Seminar Series, Report No. 9, Department of Graphics and Engineering Drawing, Princeton University, December 5, 1963.

15.

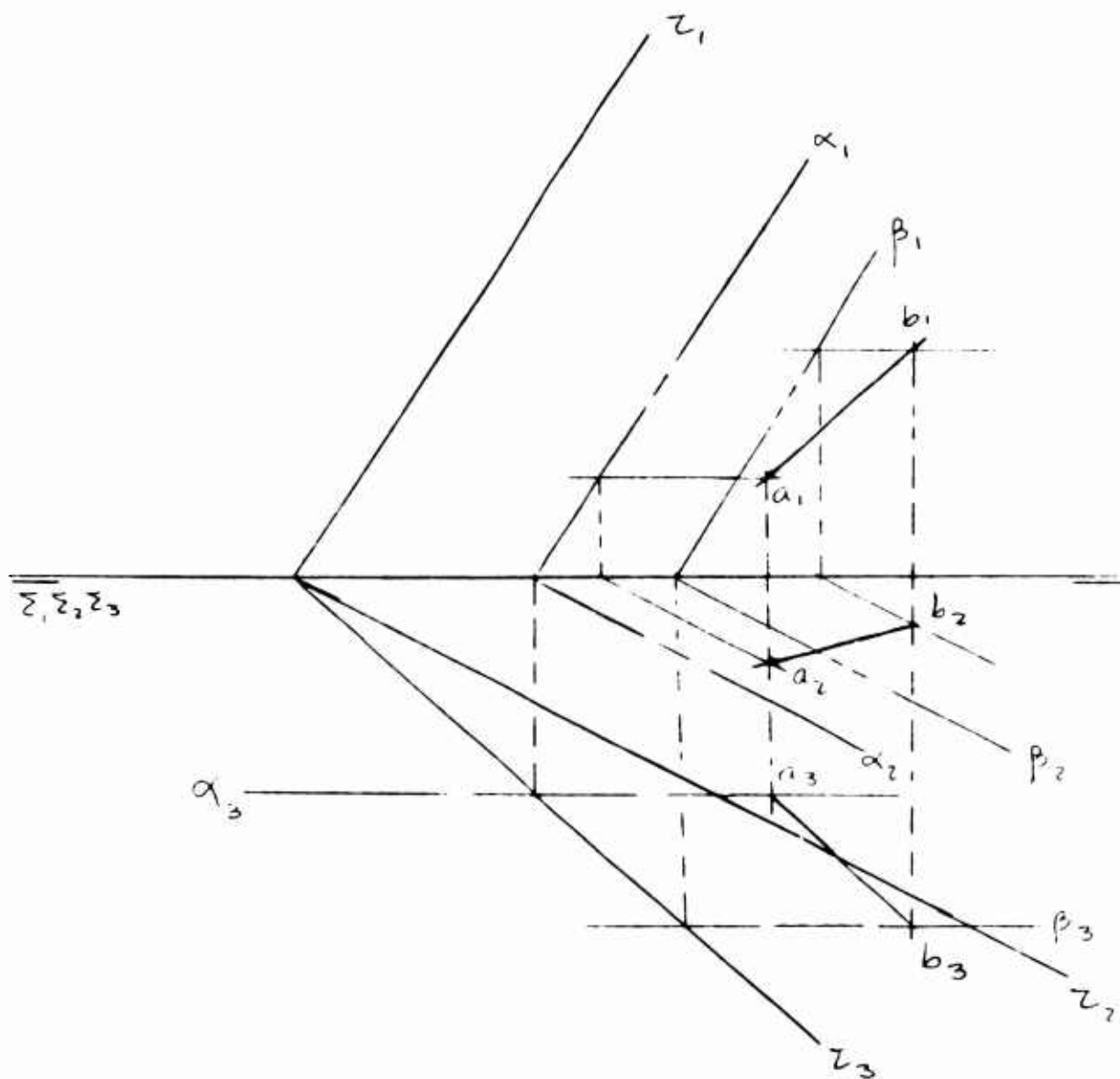


Figure 15

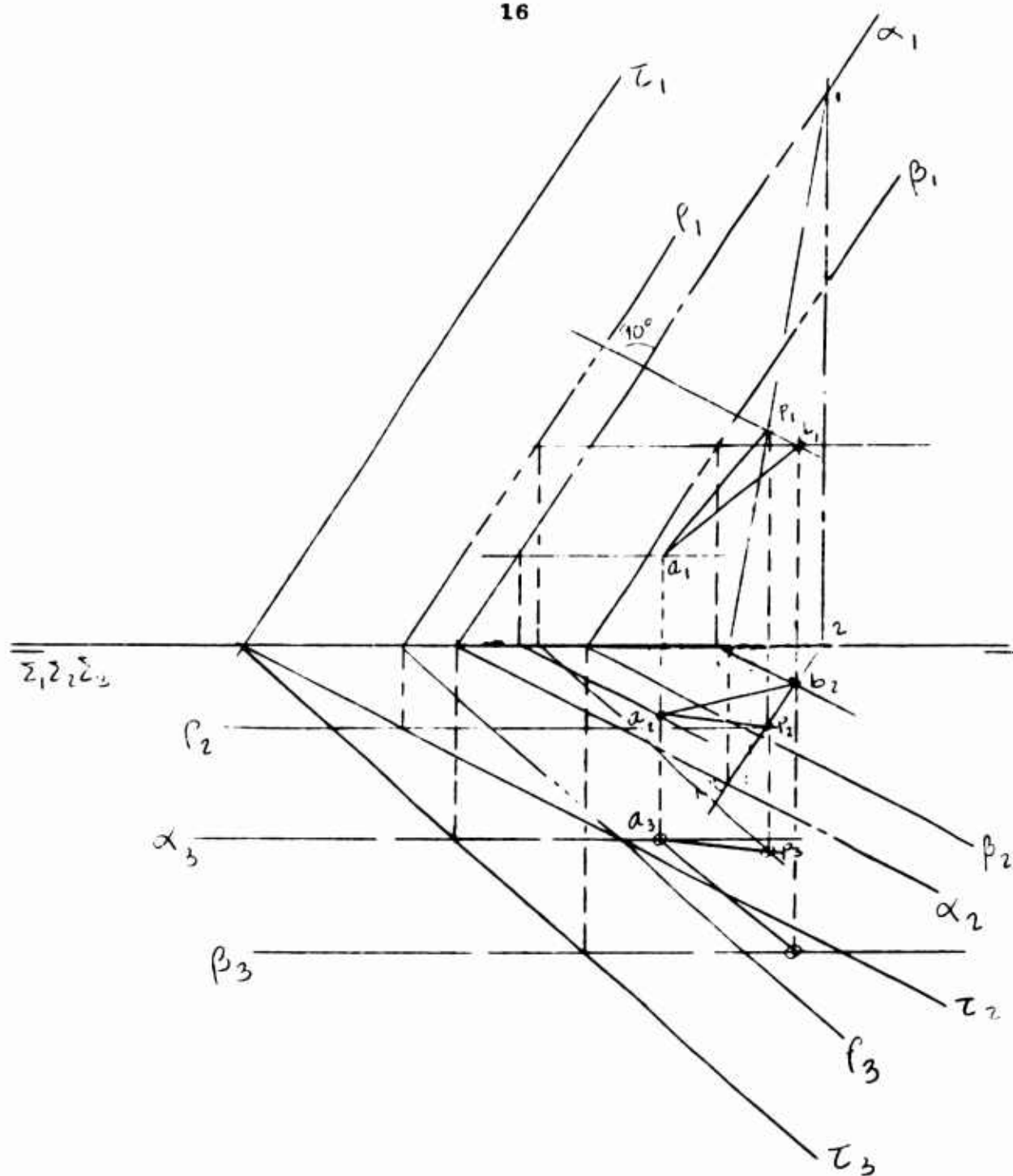


Figure 16

the point (a), and we shall determine the angle that (ab) forms with α .

Through (b) draw the perpendicular to α and determine the foot (p) of the perpendicular. (Figure 16).

To determine the angle of (ab) and (bp) apply the solution outlined for Problem No. 1.

Problem No. 3

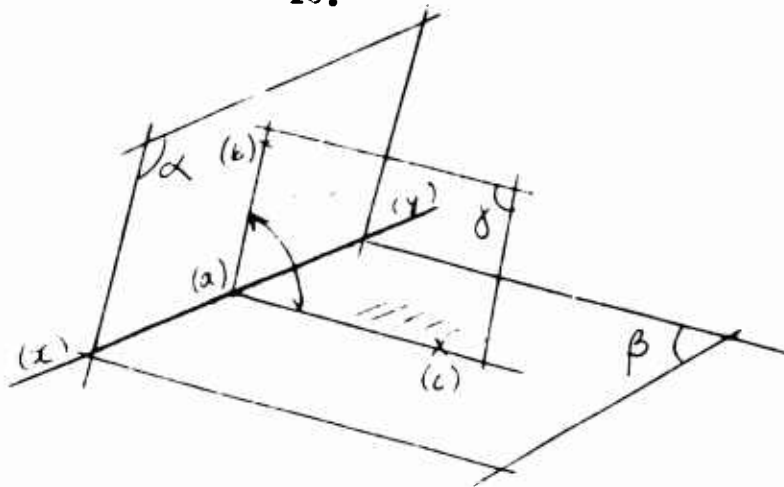
"To determine the true value of the angle of two planes that belong to the same 3-D space".

Solution

Let α and β be the given planes, intersecting along a line (xy). Through (a) of (xy) pass a plane γ , perpendicular to (xy) and determine $\alpha \times \gamma \Rightarrow (ab)$ and $\beta \times \gamma \Rightarrow (ac)$. The true value of the angle (\hat{bac}) is the solution sought. (Figure 17).

In Figure 18, the plane α is given parallel to the 3-D space Σ_3 and the plane β is given by the points (m), (n), (p). The intersection of the two planes is the line (xy).

18.



$$\alpha \perp \gamma ; \beta \perp \gamma ; \gamma \perp (xy)$$

Figure 17

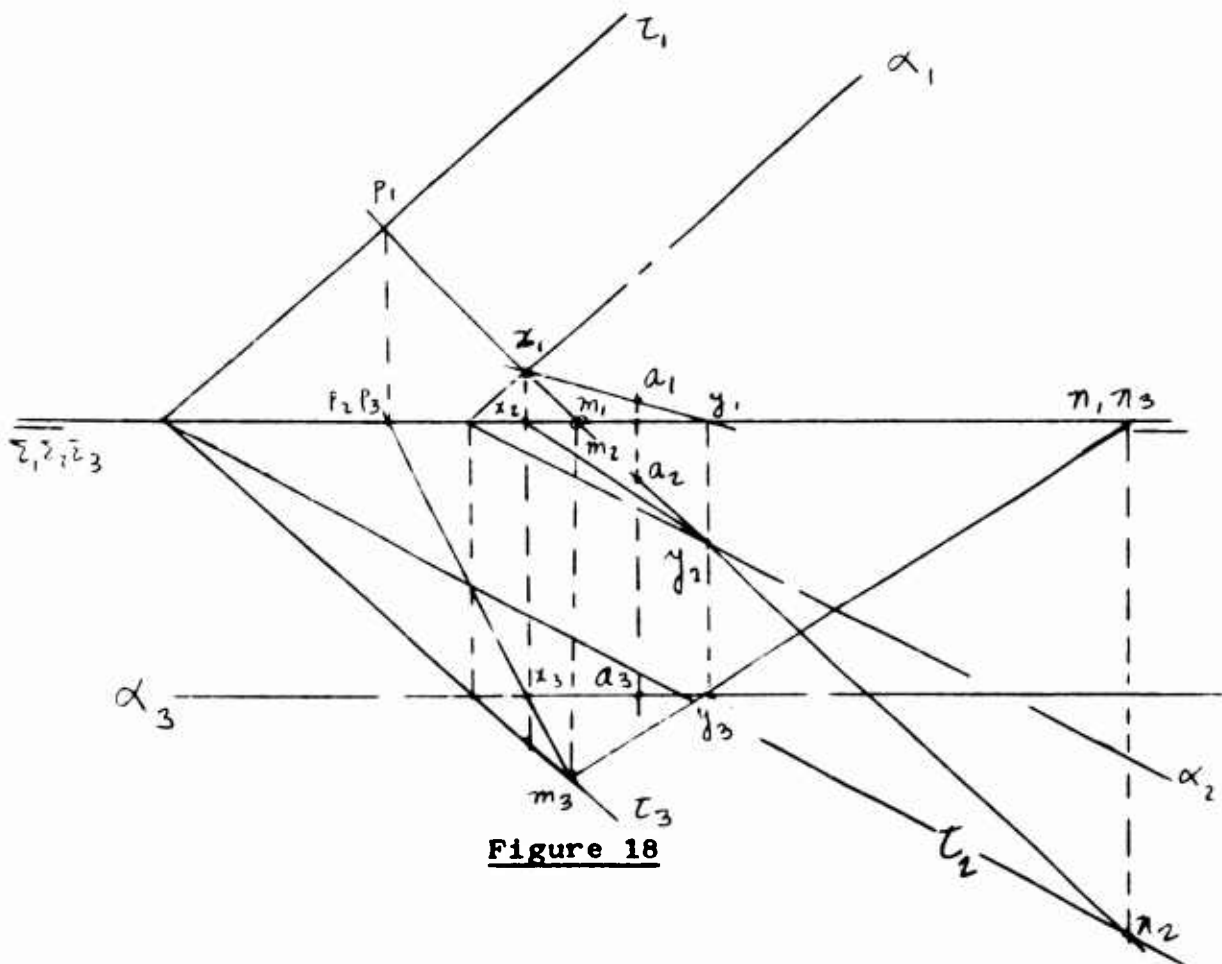


Figure 18

19.

To determine (ab) perpendicular to (xy) in plane α , apply rotation to the plane, about α_2 , until superimposition on plane π_2 of the 4-D system of reference. See Figure 19.

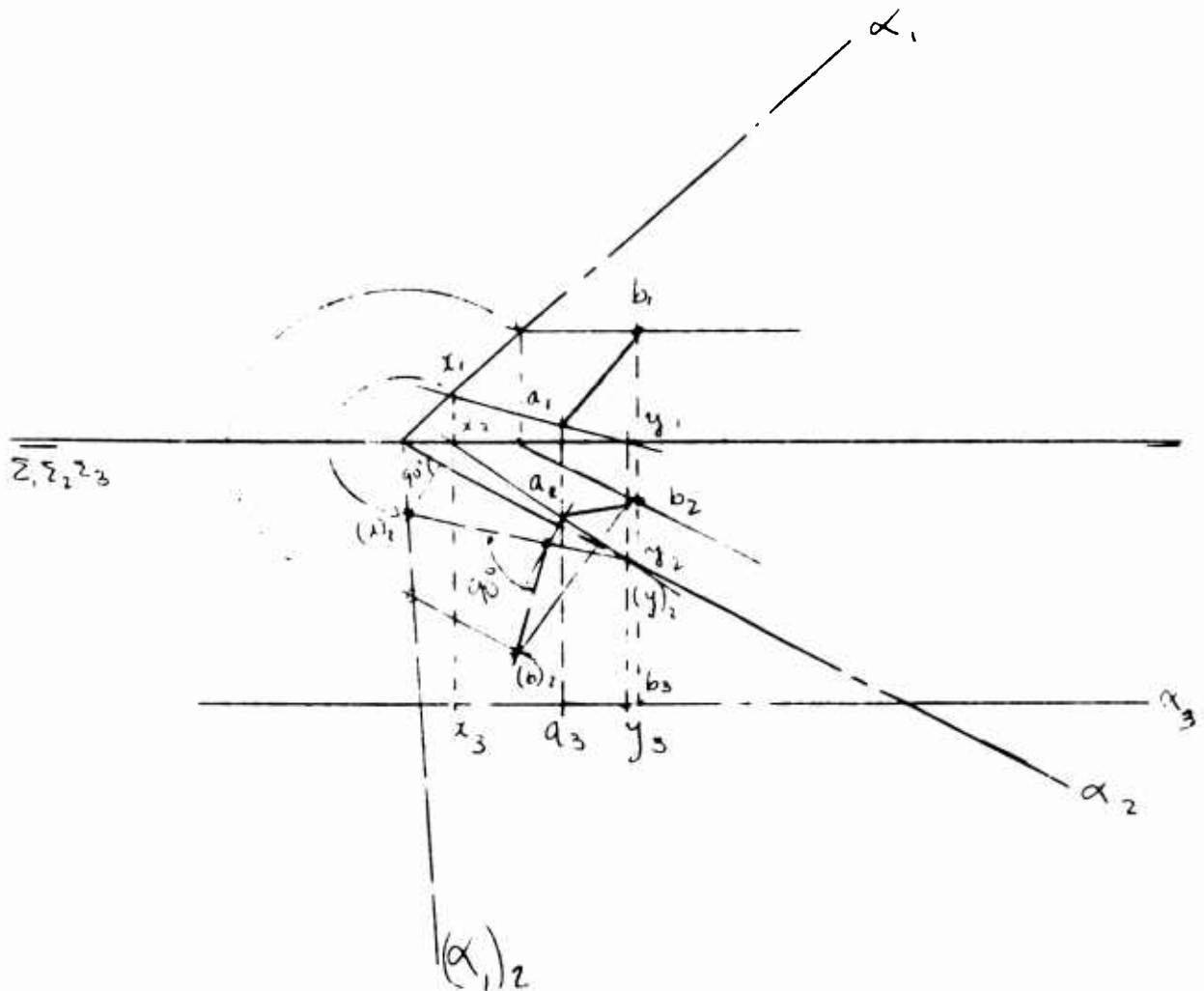


Figure 19

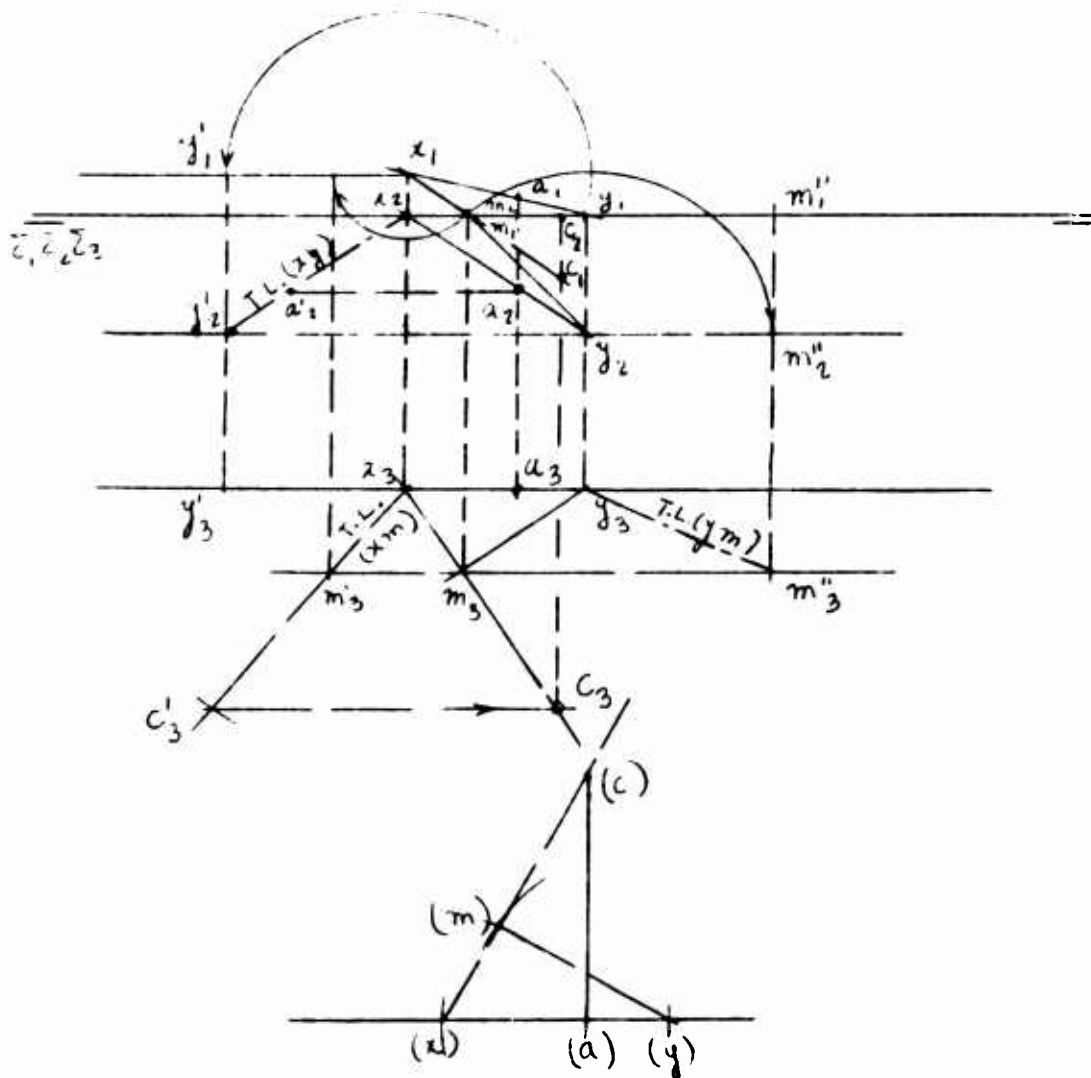


Figure 20

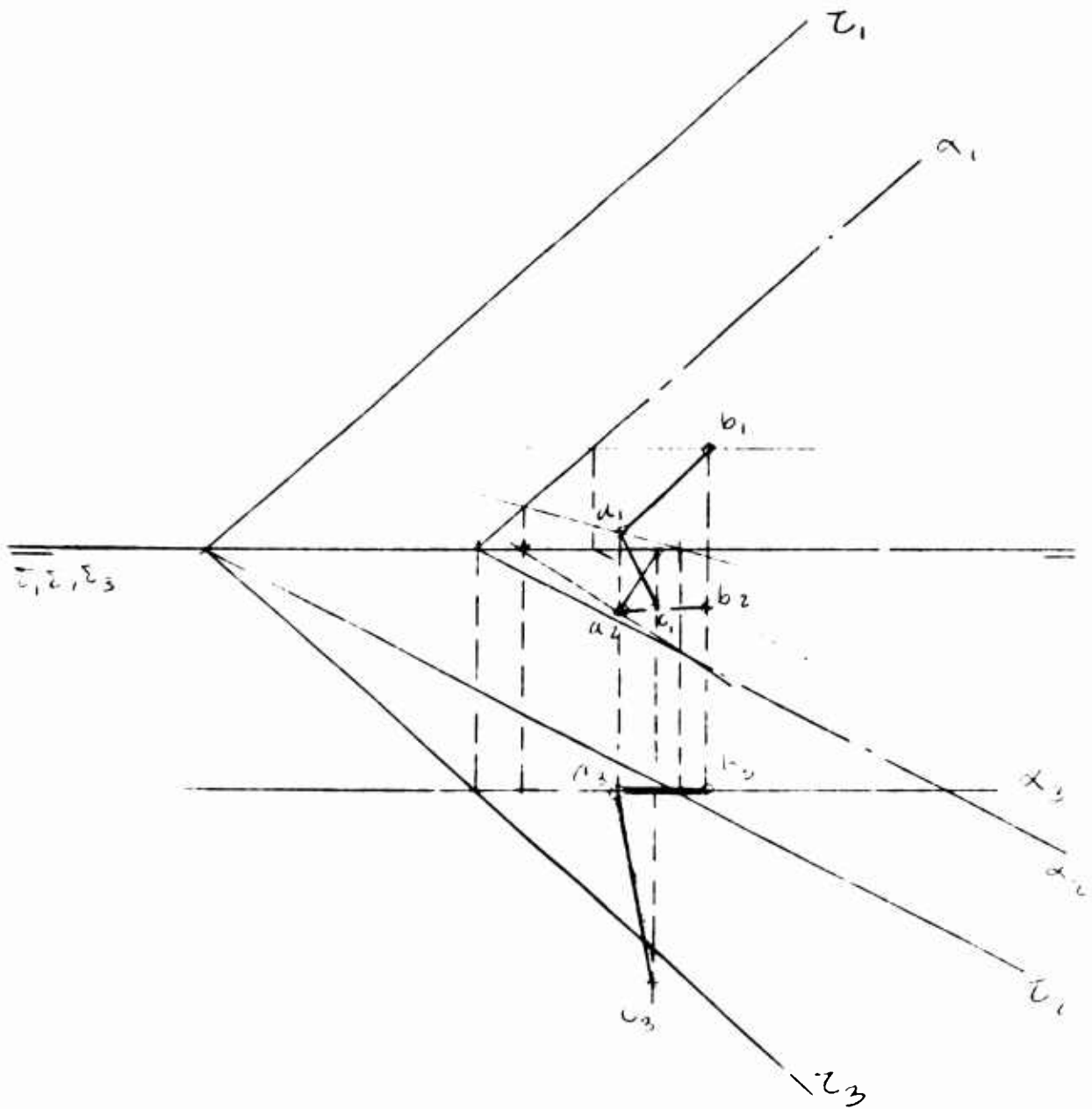


Figure 21

To determine (bc) perpendicular to (xy) , in plane $\beta \equiv (mnp)$, apply changes in the 3-D spaces of reference, until β belongs to one of them, or determine the true lengths of the sides of the triangle (xym) , draw (ac) perpendicular to (xy) , and determine the projections of (c) . See Figure 20.

Thus, we obtained the sides (ab) and (ac) of the plane angle that measures the angle of the two planes α and β (Figure 21).

Apply the solution of Problem No. 1 to obtain the true value of the angle.

Problem No. 4

"To determine the true value of the angles of two planes belonging to distinct 3-D spaces".

Solution

Let Λ and Ω be two given 3-D spaces, and α a plane of Λ , and β a plane of Ω . (Figure 22).

First determine the plane of intersection of the two 3-D spaces. That is plane $(abc) = \gamma$, Figure 23.

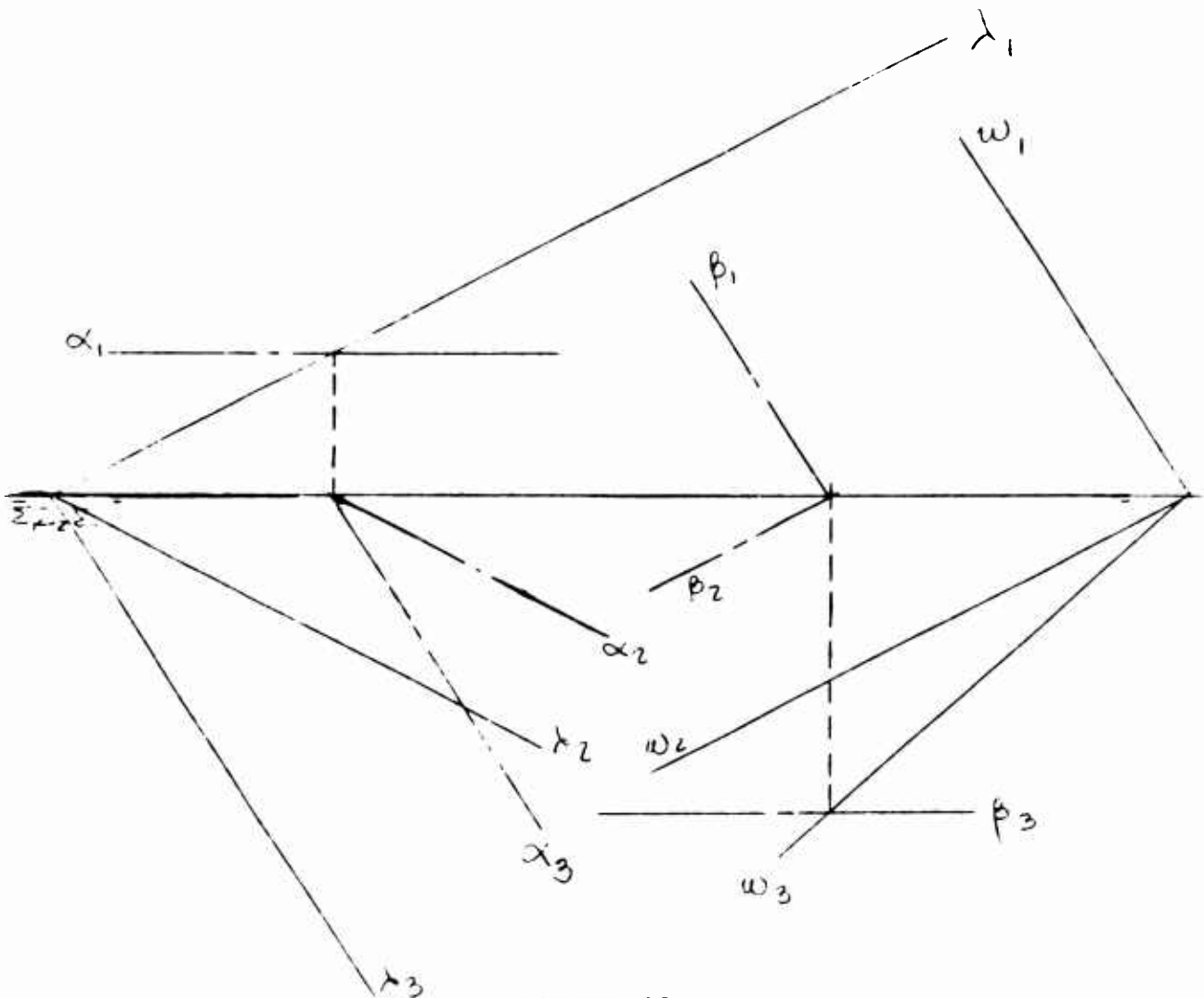


Figure 22

Next, determine the intersection of planes (abc) and α , in 3-D space Λ . That is line (mn), Figure 24.

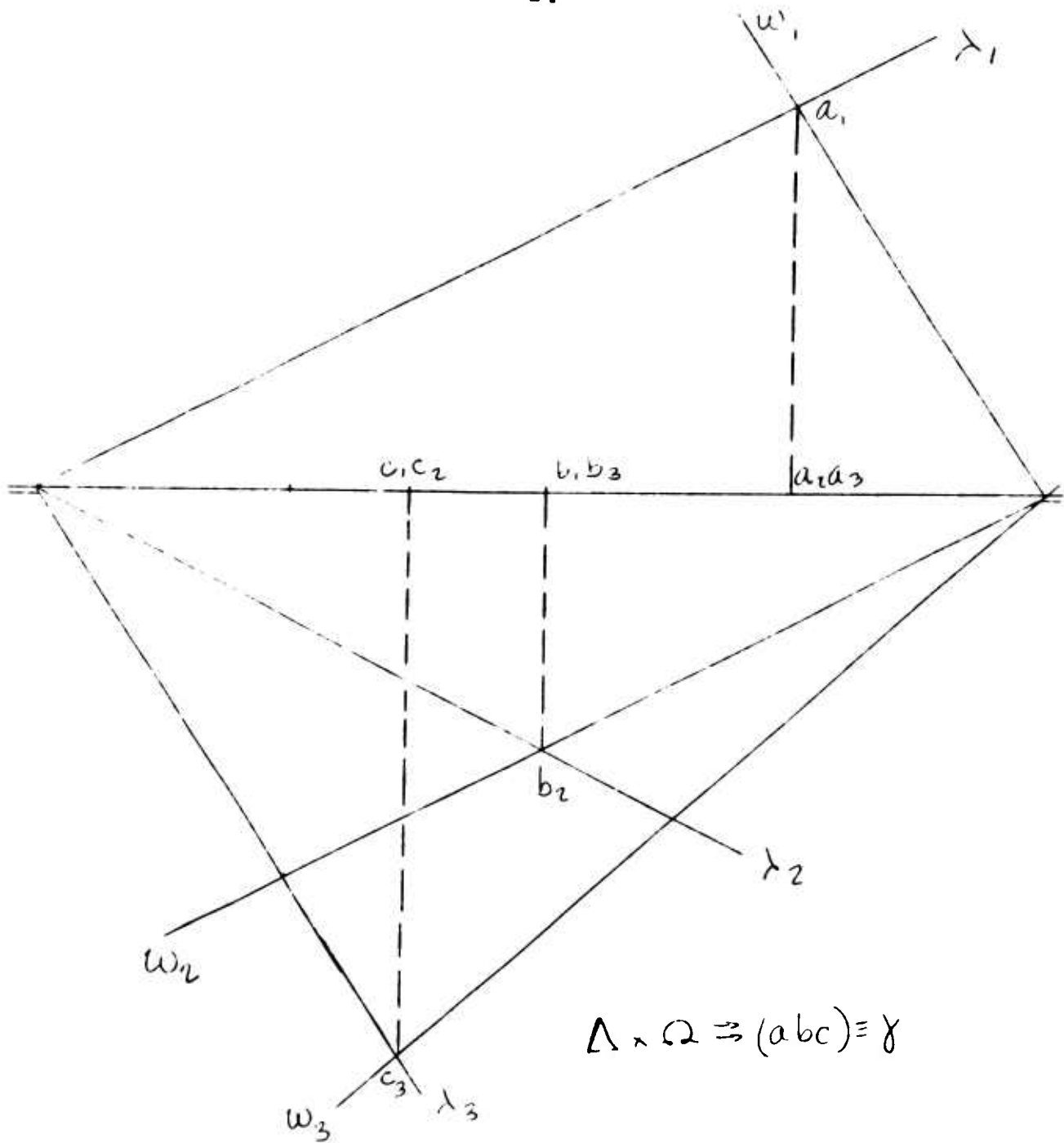


Figure 23

Next, determine the intersection of planes (abc) and β , in a 3-D space Ω . That, is line (op) , Figure 25.

Finally, determine the intersection of lines (mn) and (op), point (s), in plane (abc). (Figure 26). The point (s) is the intersection of planes α and β .

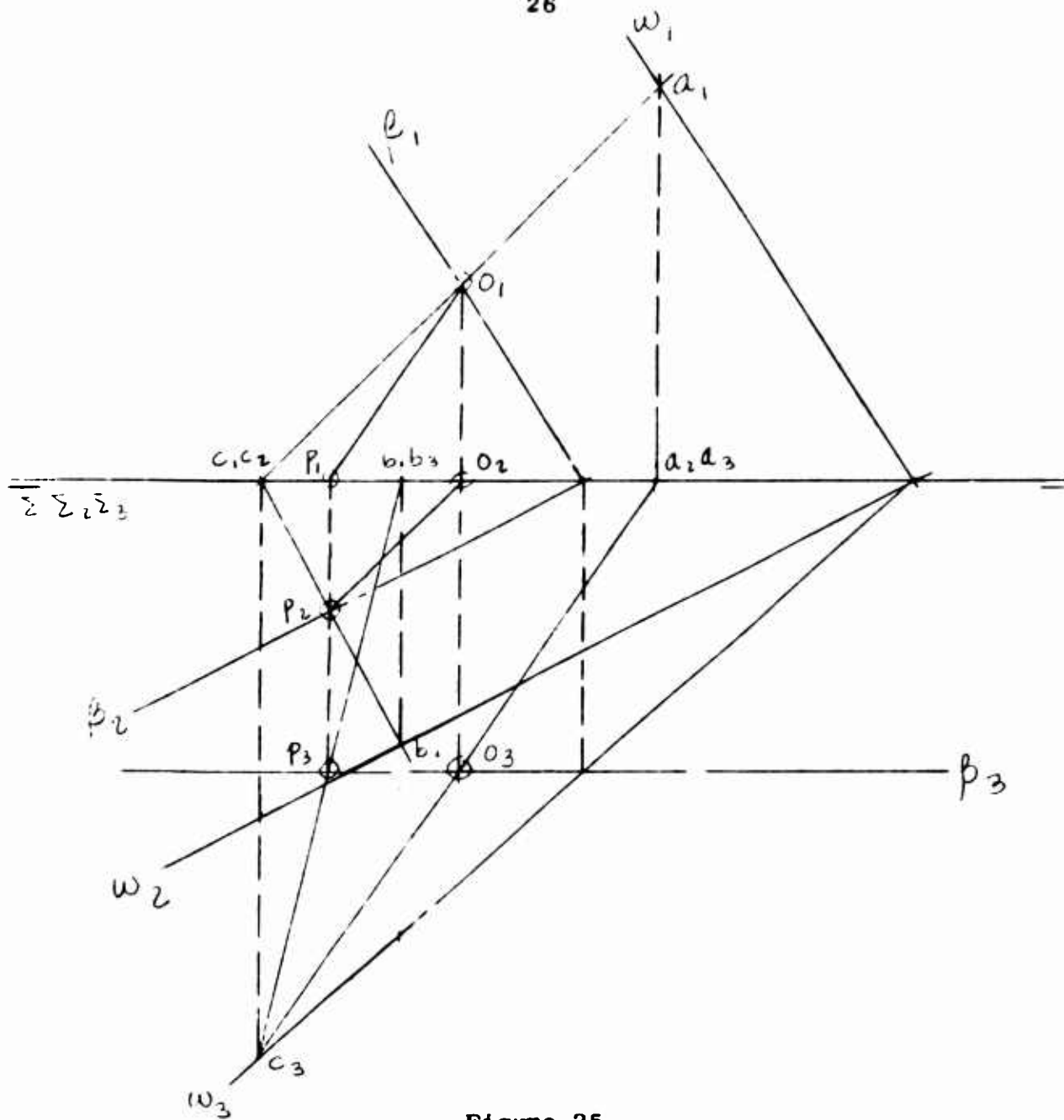


Figure 25

Figure 27 shows all the constructions just outlined.

The point (s) is the vertex of the two minimal angles between the two planes.

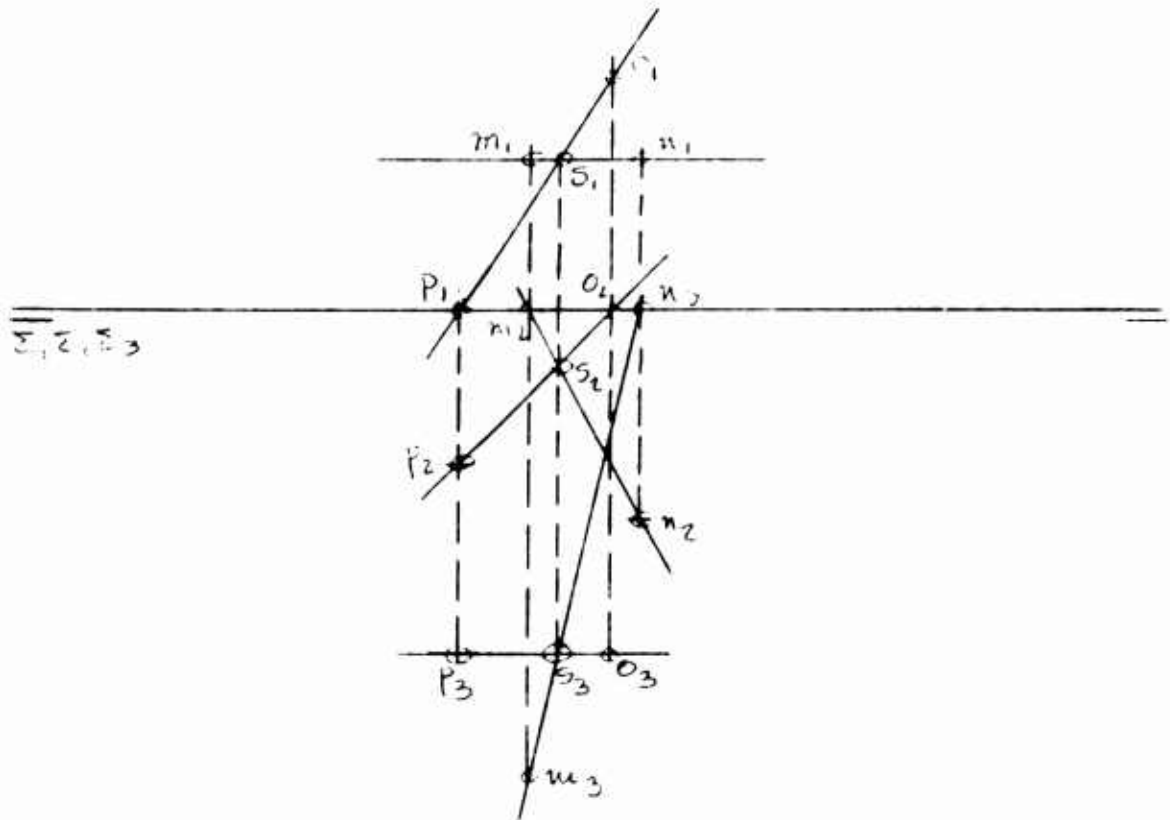
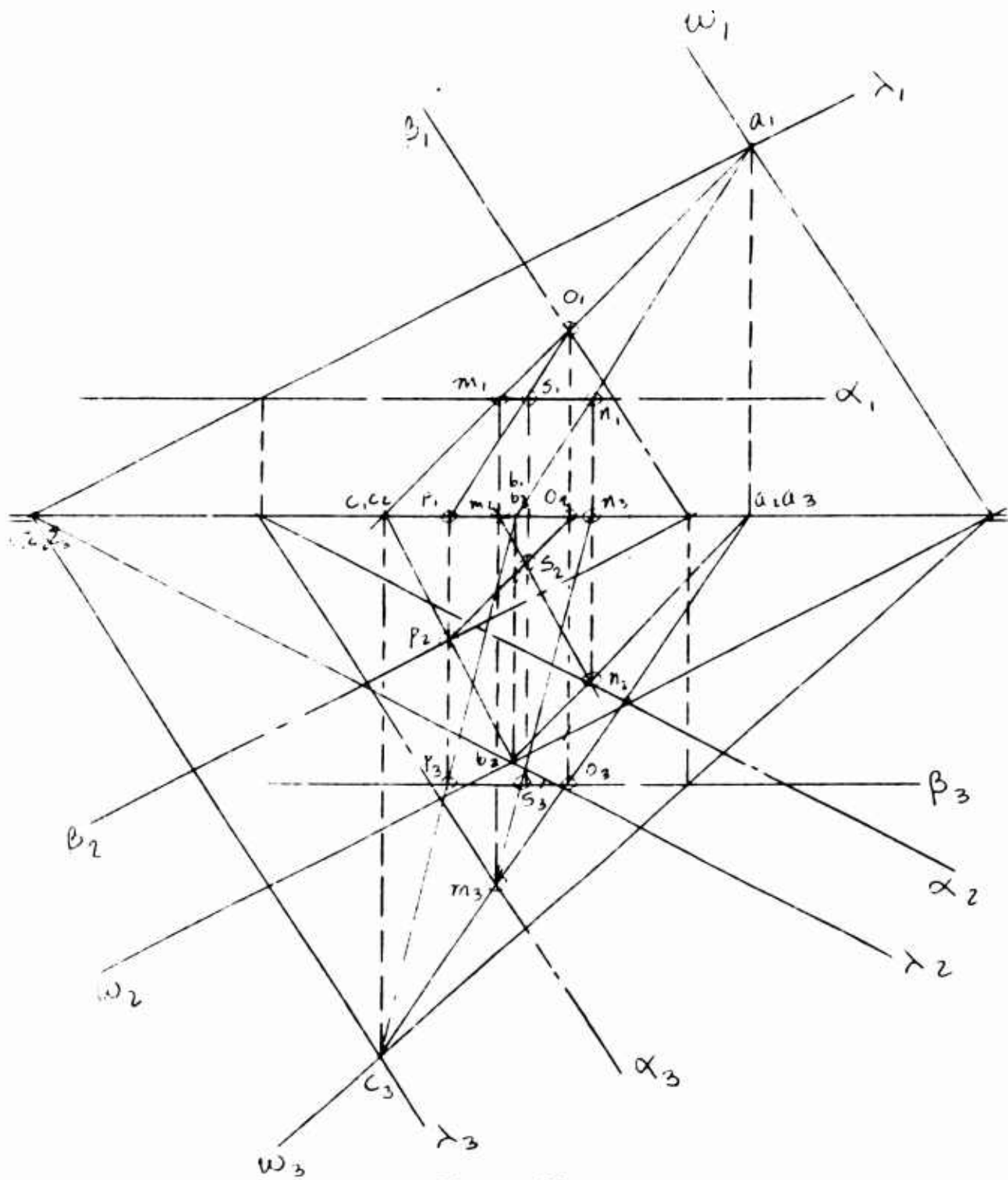


Figure 26

To determine the sides of the first angle, through
 (s) draw (st) perpendicular to plane α , in 3-D space Δ ,
 and (sv) perpendicular to plane β , in 3-D space Ω .
 (Figure 28).

**Figure 27**

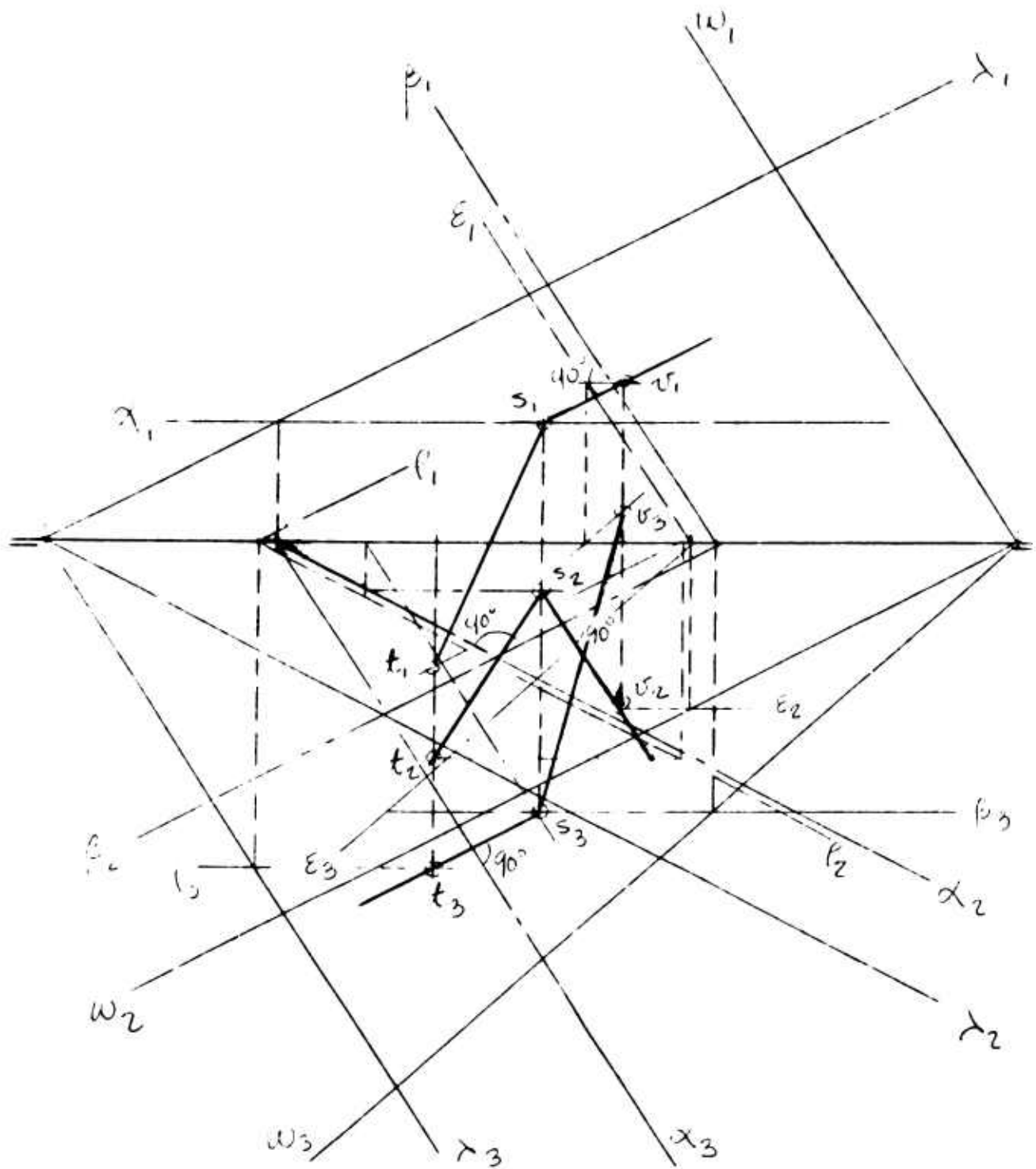


Figure 28

The first minimal angle is $\varphi = 180^\circ - (\hat{t}\hat{s}v)$

To determine the second angle, through (s) raise (sy) perpendicular to 3-D space Δ and (sx) perpendicular to 3-D space Ω . (Figure 24).

The second minimal angle is $\theta = 180^\circ - (\hat{x}\hat{s}y)$

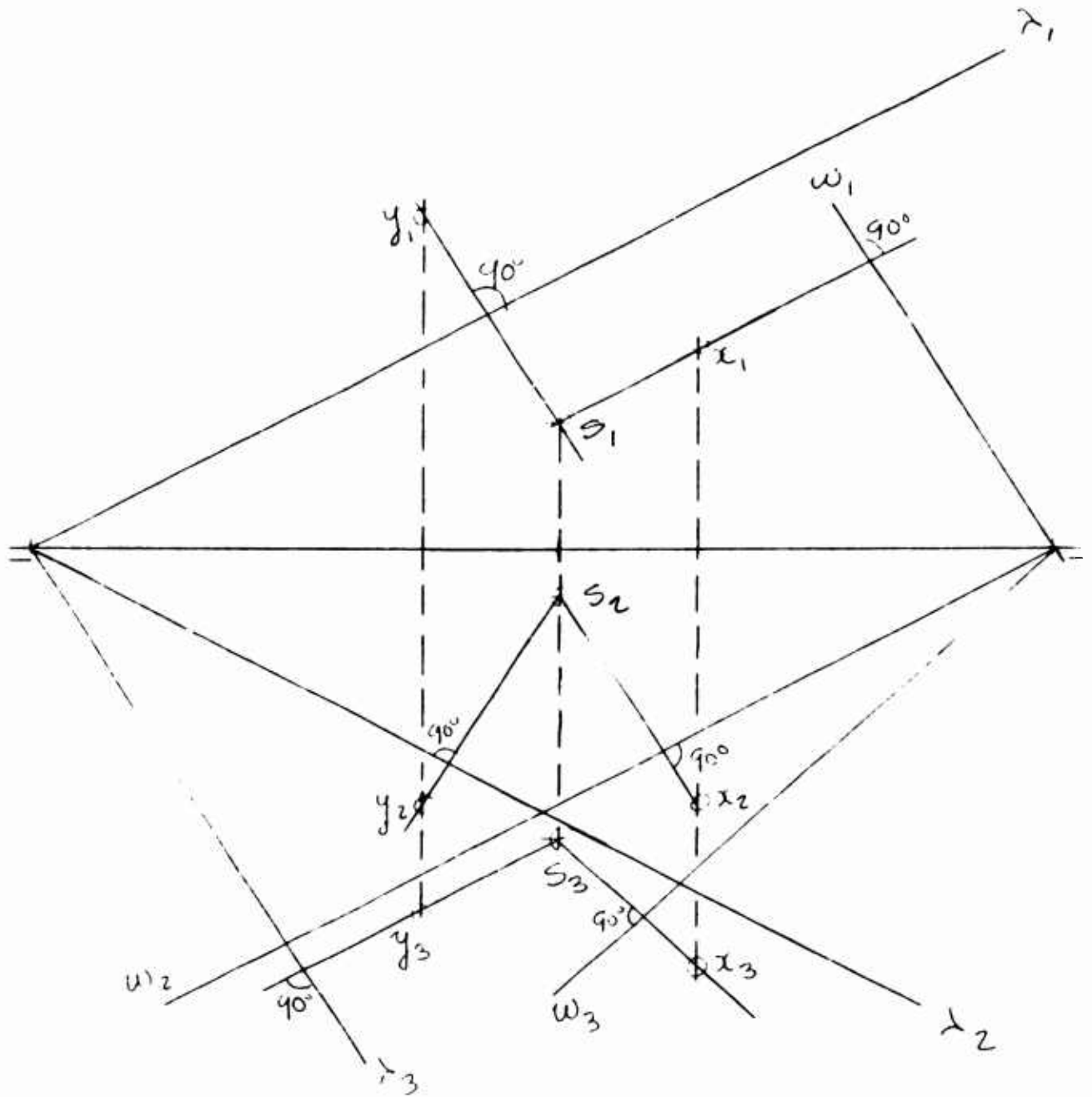


Figure 29

Problem No. 5

"To determine the true value of the angle between a line and a 3-D space".

Solution

The angle of a line and a 3-D space is equal to the angle between the line and its orthogonal projection on the 3-D space.

Given the line (ab) and the 3-D space Δ (Figure 30), through (a) and (b) draw perpendiculars to Δ . We have thus defined a plane α , perpendicular to Δ . (Figure 31).

To obtain the intersection of α and Δ (which will be the projection of (ab) on Δ), consider a 3-D space T belonging to α - Figure 32. The intersection of T and Δ is a plane β . (Figure 33). The intersection of α and β is the projection (cd) of (ab) on Δ . The two lines, (ab) and (cd) will be concurrent at a point (o), vertex of the angle. (Figures 34 and 35).

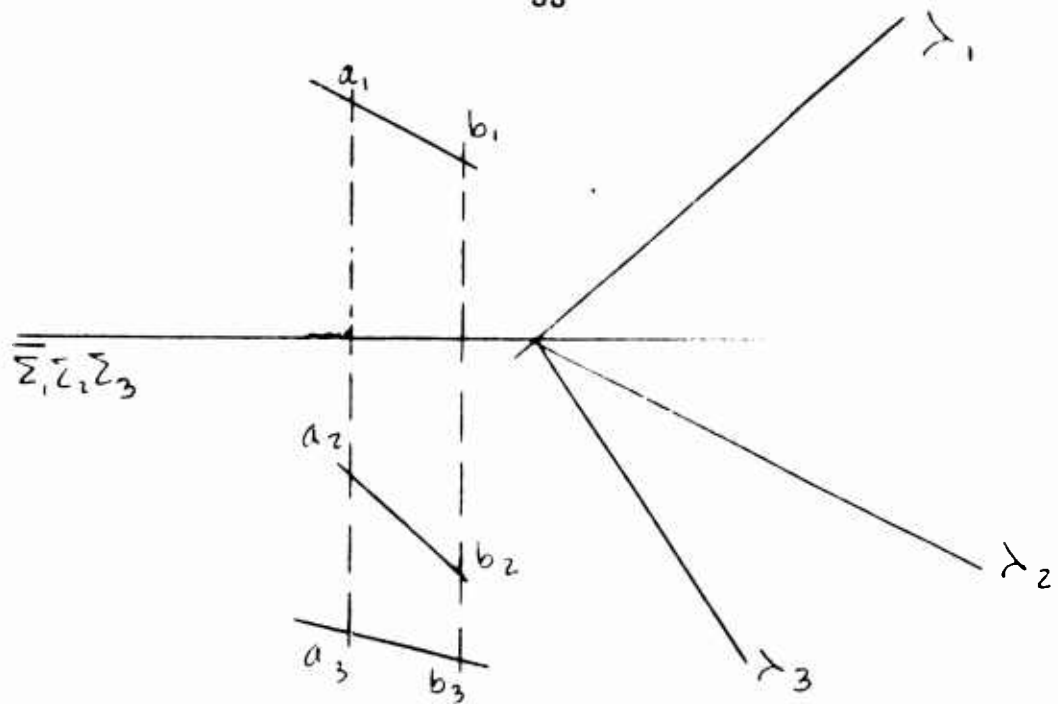


Figure 30

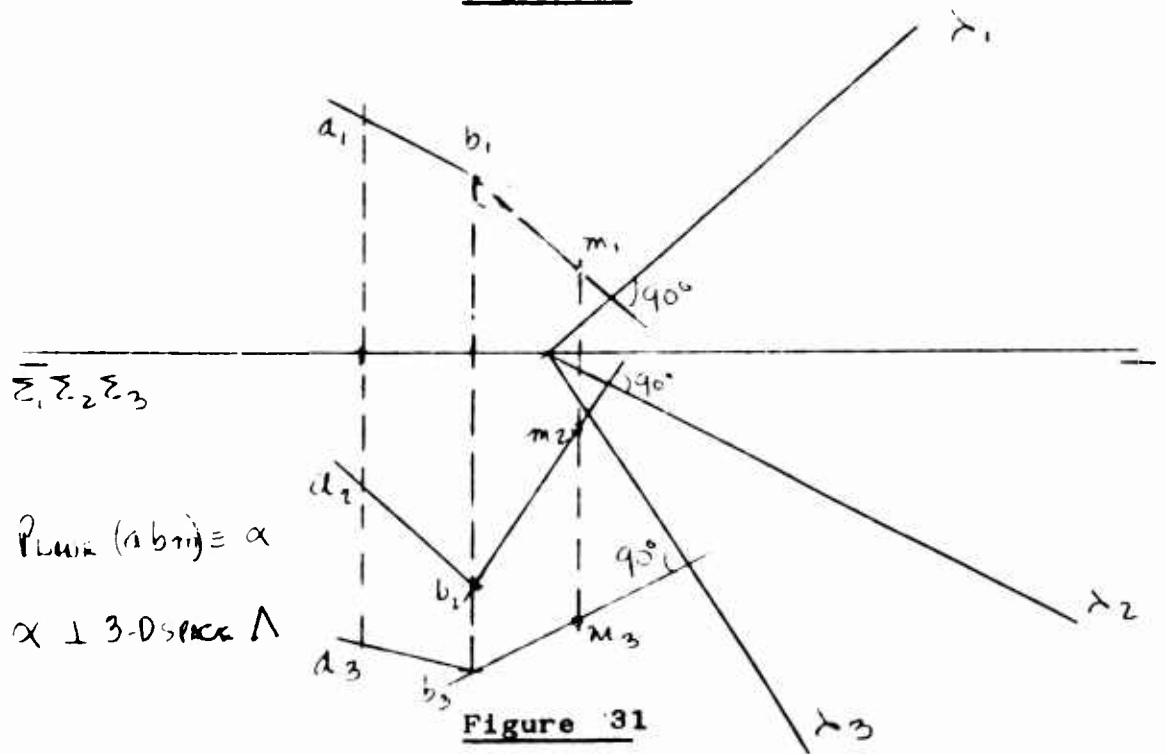


Figure 31

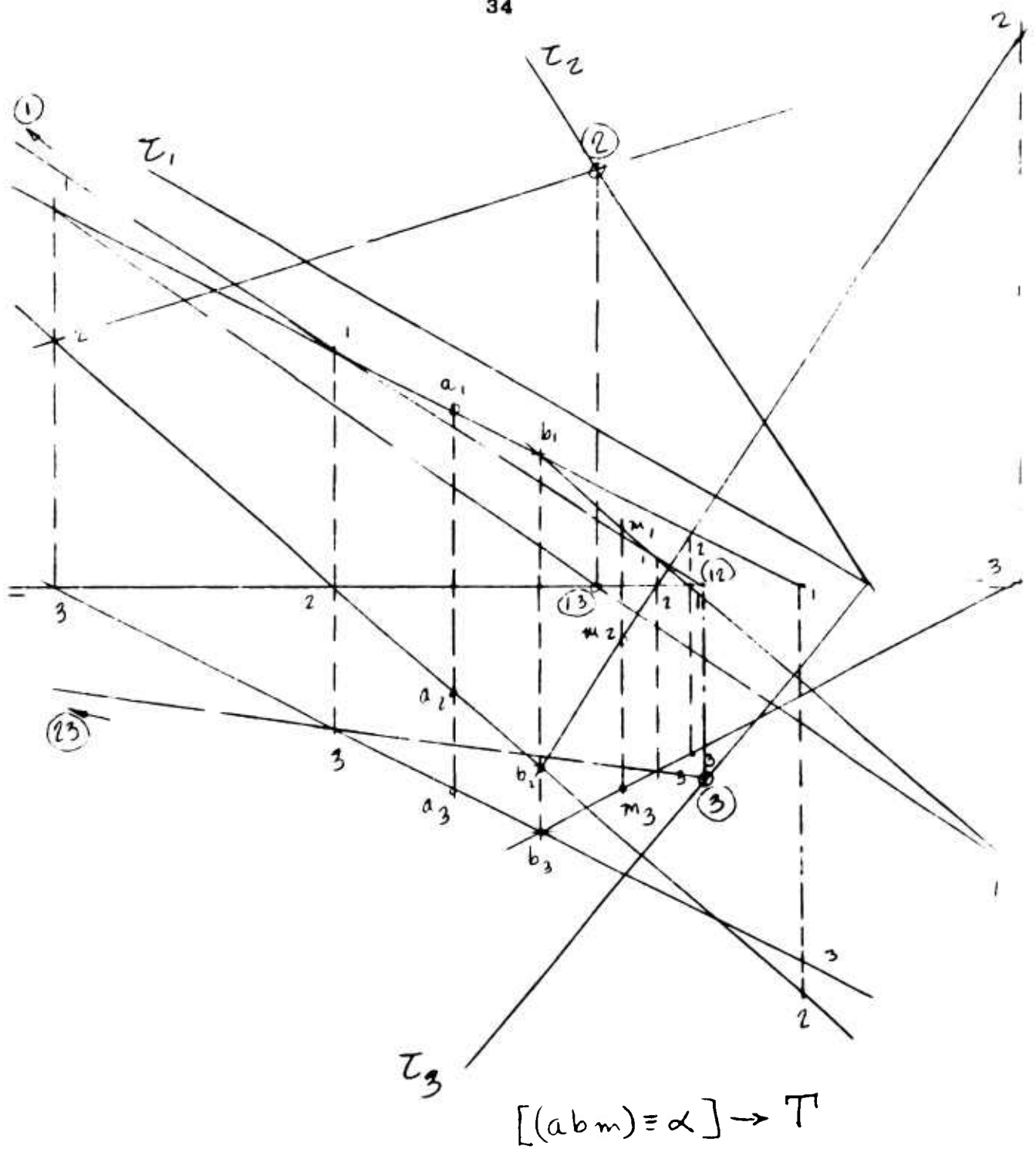
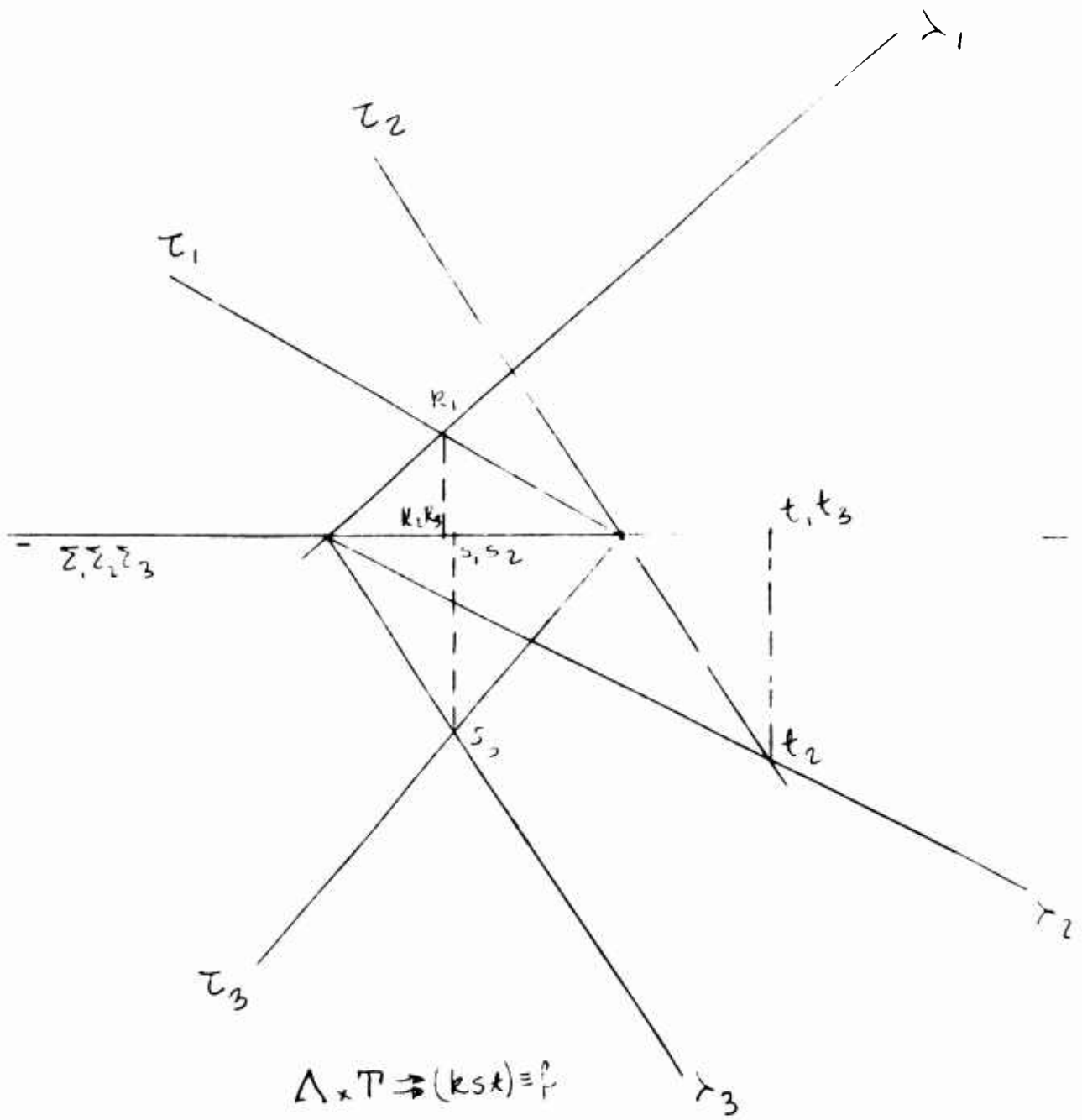


Figure 32

**Figure 33**

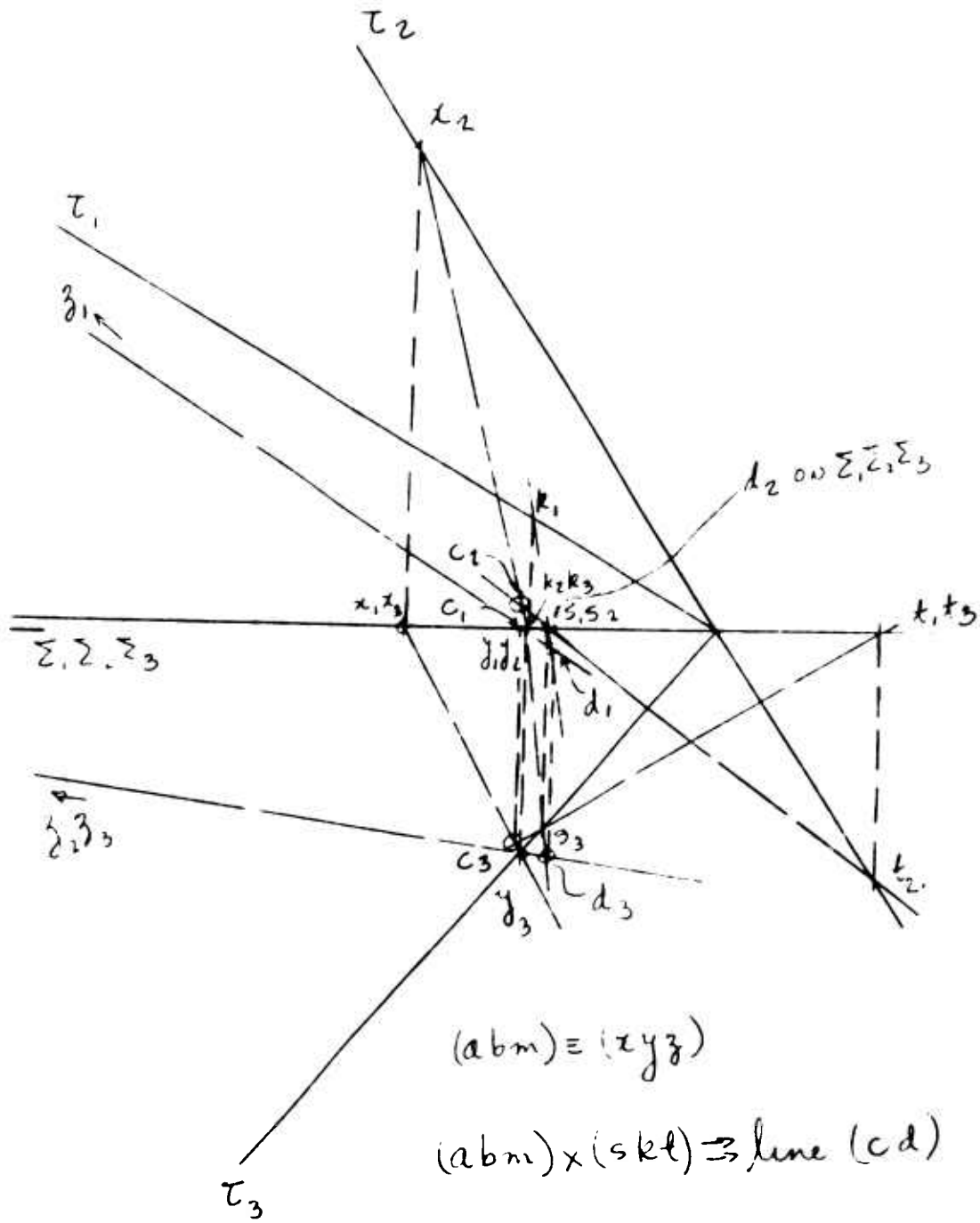


Figure 34

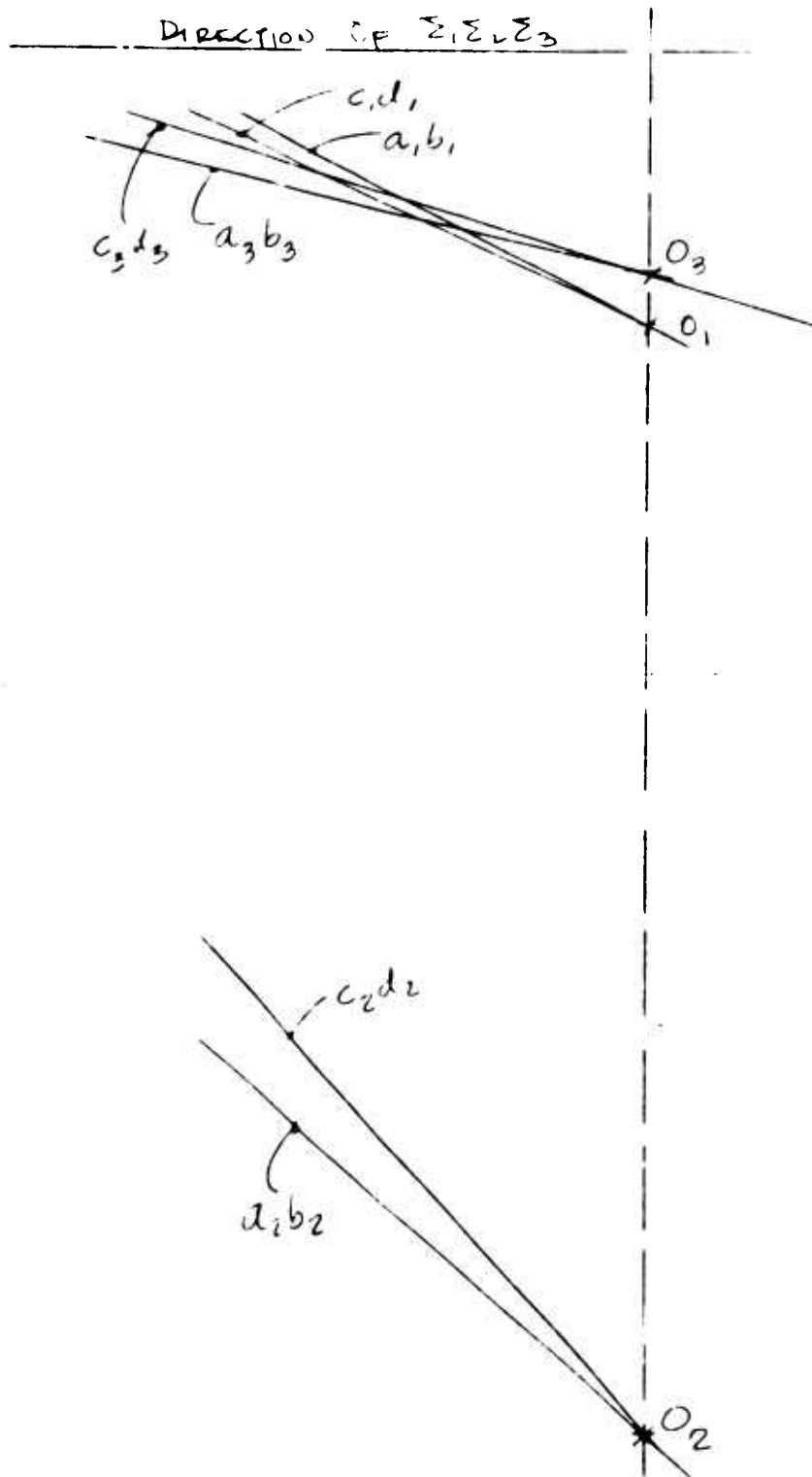


Figure 35

Problem No. 6

"To determine the angle of two 3-D spaces".

Solution

The plane angle which measures the angle of the two 3-D spaces has its vertice in the plane of intersection of the two 3-D spaces and its sides, one in each 3-D space, perpendicular to that plane of intersection.

Let Δ and ∇ be the two given 3-D spaces. Their intersection is the plane $(abc) \equiv \alpha$, on which we mark a point (o), to be vertice of a plane angle. (Figure 36).

To determine the projections of (os) of Δ perpendicular to $\alpha \equiv (abc)$ and (ot) of ∇ perpendicular to $\alpha \equiv (abc)$ it will be necessary to apply a descriptive method with one of the following objectives:

- Obtain superimposition of Δ and ∇ (by independent application of the method) with one of the 3-D spaces of the 4-D system of reference;
- Apply the method so that one of the 3-D spaces becomes parallel to a 3-D space Σ_1 of the 4-D system of reference.

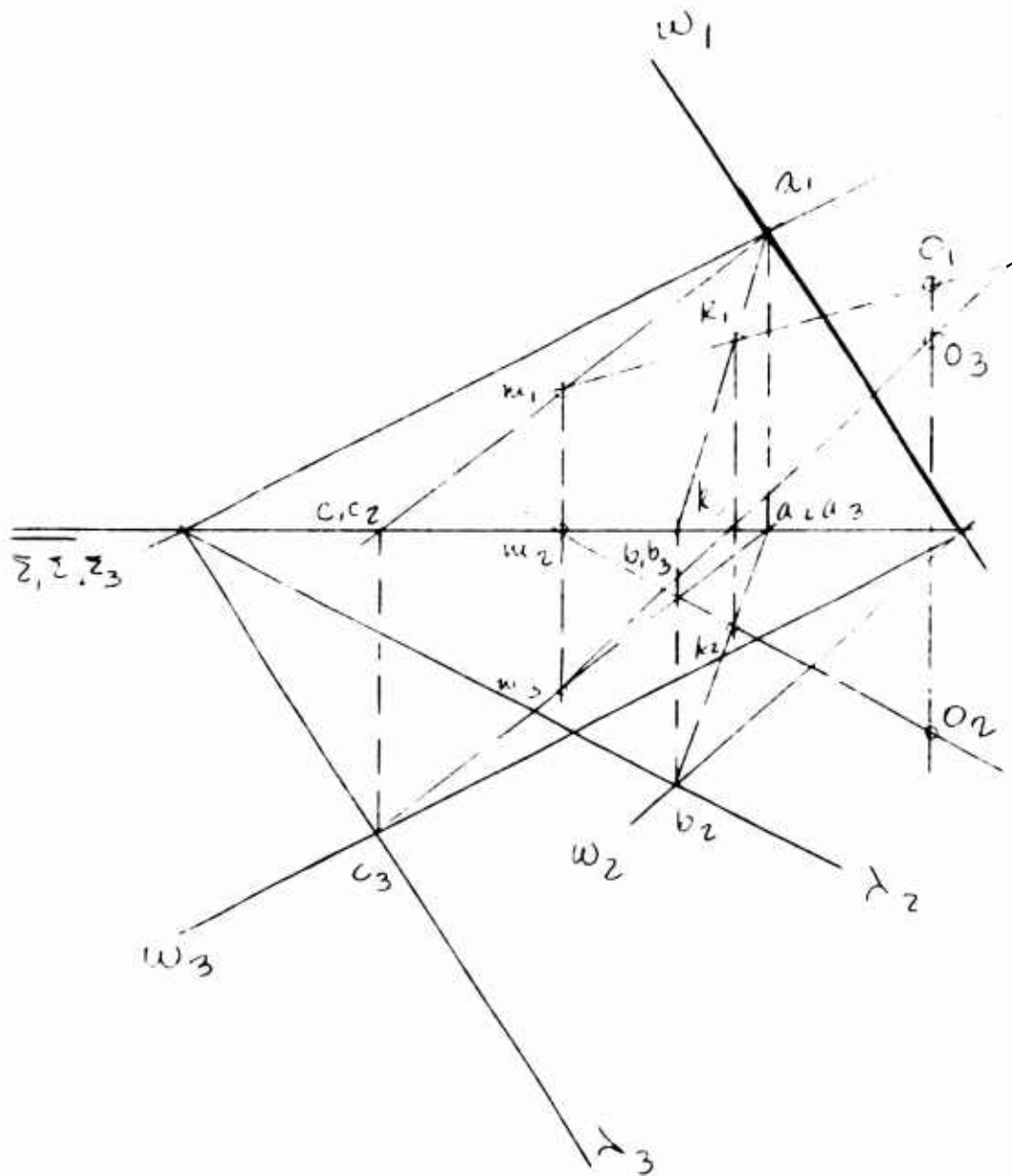


Figure 36

The result of the second alternative solution is shown in Figure 37. The 3-D space Δ is parallel to the 3-D space Σ_3 .

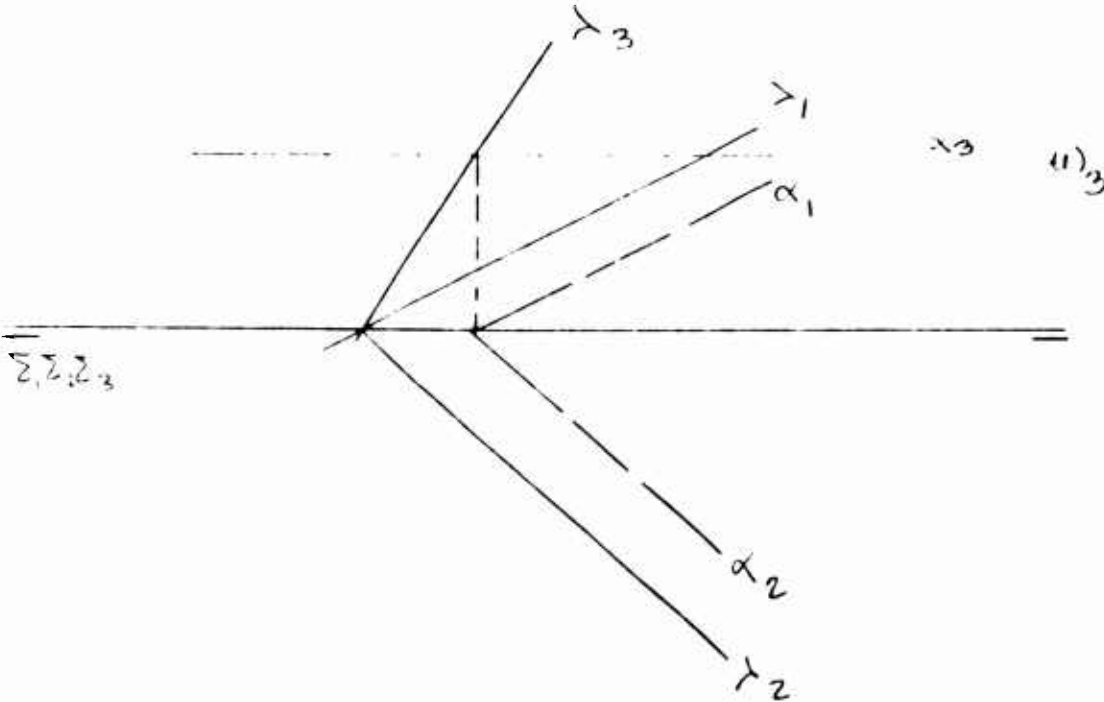


Figure 37.

In Figure 38 we show the perpendicular (os) to α , in the 3-D space Δ , and the perpendicular (ot) to α , in the 3-D space Δ . The true value of the angle $(\hat{t\hat{o}s}) = \varphi$ may be obtained by applying the solution of Problem No. 1.

41.

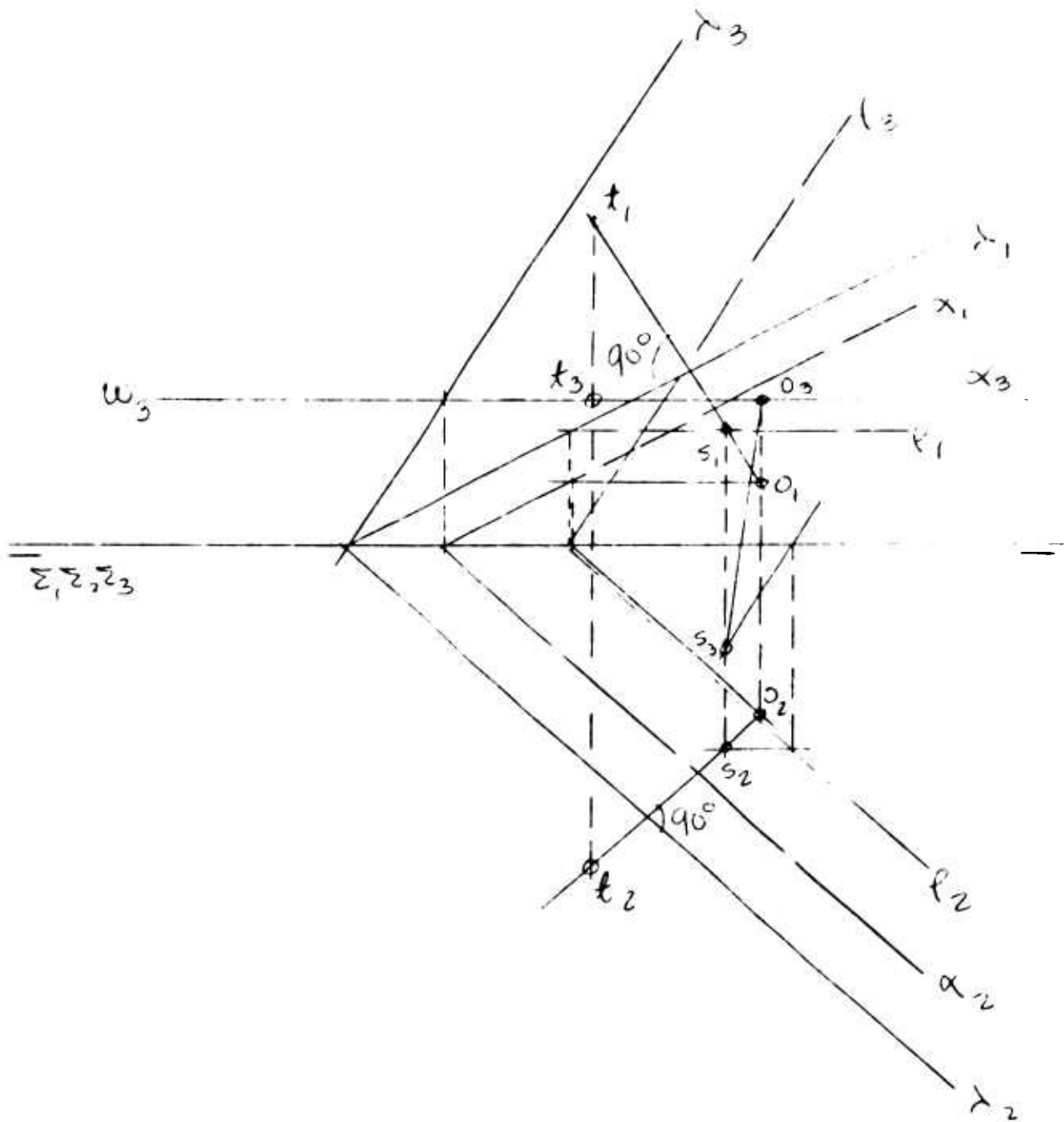


Figure 38

Problem No. 7

"To determine the true value of the angle between a plane and a 3-D space".

Solution

The angle that a plane makes with a 3-D space is the dihedral angle formed by the plane and the plane of intersection of a 3-D space belonging to the given plane and perpendicular to the given 3-D space.

Let (abc) be the given plane and τ the given space.
(Figure 39).

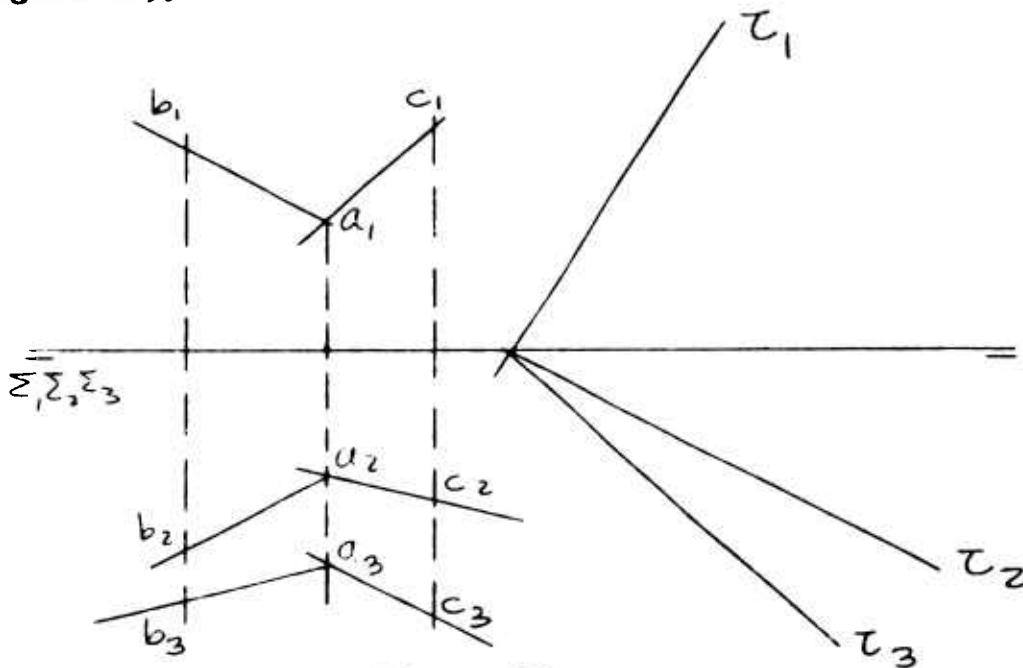
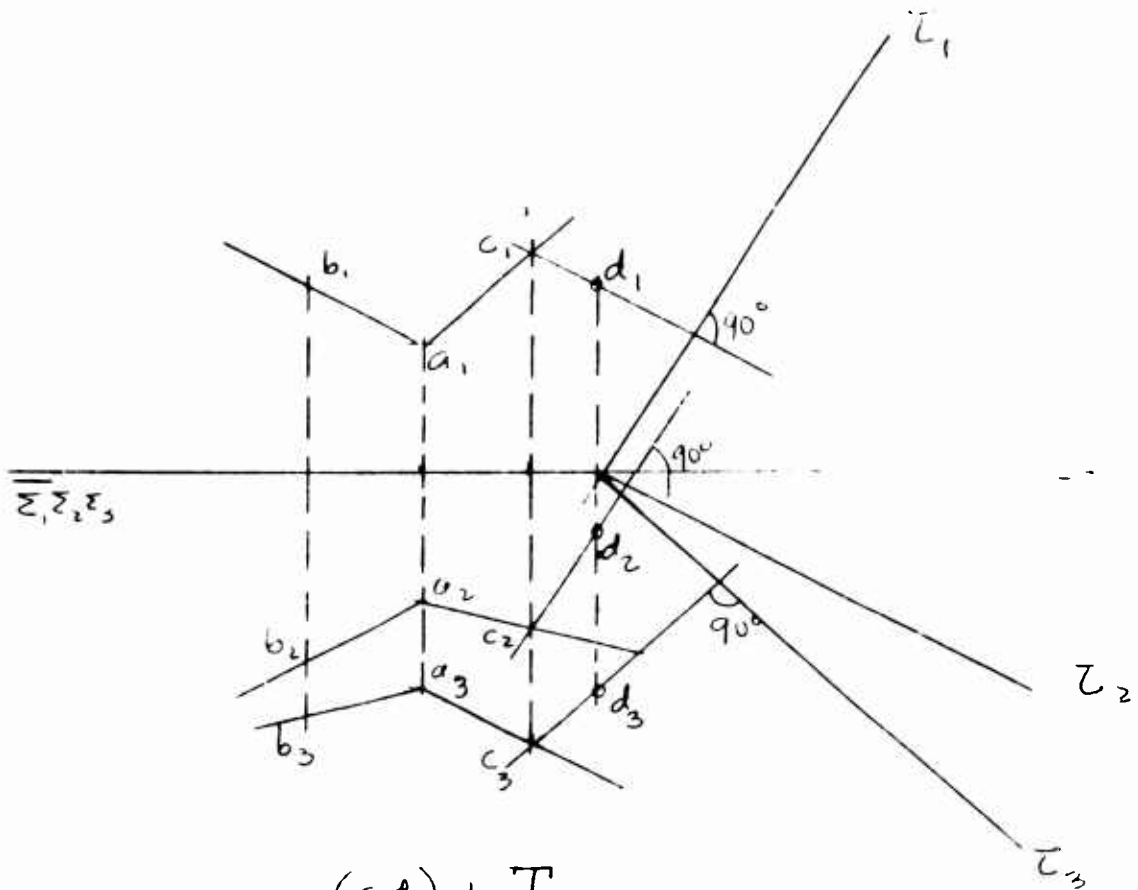


Figure 39

43.

To define a 3-D space Ω perpendicular to T , and belonging to (abc) , through a point of the plane draw a perpendicular to T . The plane (abc) and the perpendicular determine Ω . (Figure 40).



$$(cd) \perp T$$

$$[(abcd) \equiv \Omega] \perp T$$

Figure 40

To complete the problem, proceed as follows:

- Determine the intersection of Ω and \mathbb{T} , plane β .
- Determine the angle between planes α and β , concurrent along a line, since both belong to the same 3-D space, Ω .